1992

Inference procedures in some lifetime models.

Kulathavaranee. Thiagarajah
University of Windsor

INFERENCE PROCEDURES IN SOME LIFETIME MODELS

by

KULATHAVARANEE THIAGARAJAH

A Dissertation
submitted to the Faculty of Graduate Studies and Research
Through the Department of Mathematics and Statistics
in Partial Fulfillment
of the requirements for the Degree of
Doctor of Philosophy
at the University of Windsor

Windsor, Ontario, Canada
1992

ISBN 0-315-78903-4

Respectfully Dedicated to my Mother, and
to the Loving Memory of my Father

ABSTRACT

This thesis deals with inference procedures for some parametric lifetime models, involving single as well as multiple samples. In some situations censored (Type I and Type II) samples are considered. The thesis consists of two parts. Part I deals with homogeneity testing involving multiple samples from the gamma, exponential and the Weibull or the extreme value distributions. Part II deals with confidence interval procedures for the parameters of the two parameter exponential distribution and the extreme value models.

Assuming the underlying distribution for several groups of data to be two parameter gamma with common shape parameter various tests are developed for comparing the means of the groups. The performance of these test statistics are determined in terms of level and power by conducting simulations. A $C(\alpha)$ test and a likelihood ratio test are presented and compared for checking the validity of the assumption of common shape parameter. Under failure censoring, various test statistics for comparing the mean life times of several two parameter exponential distributions are derived and studied by performing Monte Carlo simulations.

Considering failure censored data, homogeneity tests for extreme value location parameters with the assumption of a common scale parameter are studied. For this problem, a $C(\alpha)$ test is derived and compared with other existing methods through simulations. Also, for testing the assumption of common extreme value scale parameter, a $C(\alpha)$ statistic is derived and compared with other existing statistics.

In single sample situations several confidence interval estimation procedures for the scale parameter of a two parameter exponential distribution under time censoring are discussed. Behaviours of the confidence intervals based on these procedures are examined by simulation study in terms of average lengths, coverage and tail probabilities.

For extreme value failure censored data (with or without covariates), a simple method using orthogonality approach (Cox and Reid, 1987) to obtain explicit expression for the variance-covariance of the MLEs of the parameters is given. For obtaining confidence intervals for the parameters of interest various procedures, such as the procedure based on the likelihood ratio, the procedure based on the likelihood score corrected for bias and skewness and the procedure based on the likelihood ratio adjusted for mean and variance, are derived. The behaviours of these procedures are investigated in terms of average lengths, coverage and tail probabilities by conducting Monte Carlo simulations. The above procedures are extended to extreme value regression model. Confidence interval procedures are also derived and studied for the parameters of the extreme value model under time censoring.

ACKNOWLEDGEMENTS

I wish to express my gratitude and appreciation to Dr. S. R. Paul for his constant help and guidance during the preparation of this dissertation. I would like to thank Dr. V. T. Farewell, Dr. D. Tracy, Dr. K. Fung and Dr. N. Singh for their useful comments and suggestions which greatly improved this dissertation.

I sincerely thank the Department of Mathematics and Statistics, and the University of Windsor, for giving me the financial support in terms of teaching assistantships and scholarships. The financial support from Dr. S. R. Paul through his NSERC research grant is gratefully acknowledged. I also thank Dr. R. Caron, chairman, Department of Mathematics and Statistics, University of Windsor, for providing the computer facilities for the production of this manuscript.

I am very grateful to Mrs. Jaya Sreedharan for her untiring help, encouragement and affection during my studies. My thanks are also due to all my friends specially Fotini Labropulu and Iqbal Husain for their help throughout my studies.

Finally, I express my sincere gratitude to my mother, sisters and brother, whose love, encouragement and support were vital to the completion of this dissertation.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The importance of parametric models for the analysis of life time data is well-known (Lawless, 1982; Nelson, 1982; Mann, Schefer and Singpurwalla, 1974). Some life time models, namely the gamma, exponential, Weibull or the extreme value distribution, play an important role in the analysis of reliability, survival and other related studies. Gamma distribution is widely used in weather analysis (Neyman and Scott, 1965), in hydrology (Mielke and Johnson, 1974), and in life testing and survival analysis (Lawless, 1982). In life testing problems, the exponential distribution describes the life of insulating oils and fluids, and certain items such as electronic components, light bulbs etc. (Nelson, 1982; Lawless, 1982), and in survival studies it represents the remission times in chronic diseases. Weibull distribution is widely used as a distribution for product characteristics such as strength, resistence etc. in the accelerated life tests. It is also used for the life of roller bearing, electric components, ceramics, capacitors in accelerated tests (Nelson, 1990). This distribution represents the remission times in the survival studies dealing with specified fatal diseases. Weibull data are conveniently analysed in terms of the extreme value distribution. Thus, like the Weibull distribution, the extreme value distribution is useful in a great variety of applications, particularly as a model in the study of breaking strengths, floods, manufacturing and naval engineering. Its application to air pollution problems is discussed by Singpurwalla (1972).

Parameter estimation and hypothesis tests involving single as well as multiple complete or censored samples coming from the parametric survival models are major aspects of any statistical analysis of life time data. A substantial literature is available concerning interval estimation and hypothesis testing for the underlying parameters of interest. Exact inference procedures for censored samples are often impractical. Various large sample inference procedures are presented in this thesis. The thesis consists of two parts. Part I (chapters 3 to 5) deals with homogeneity tests involving multiple samples from the above distributions. Part II (chapters 6 to 8) is concerned with various approximate interval estimation procedures for the parameters of the exponential and the extreme value distribution. In this both the two parameter and the regression models are considered. These procedures are applicable not only in lifetesting and reliability but also in medical research and other relevant areas.

In chapter 2, we review some basic concepts and the life time models, namely the gamma, the exponential, the Weibull or the extreme value distributions. Large sample hypothesis testing procedures such as the likelihood ratio (LR) test and the $C(\alpha)$ test are discussed, in general, for homogeneity tests. Approximate interval estimation procedures based on LR and the procedure based on likelihood score are also described briefly.

In chapter 3, we discuss various procedures for testing equality of $(L \geq 2)$ gamma distribution parameters. A simulation study is conducted to compare the performance of the test statistics in terms of size and power. In view of the complicated expressions of the quantiles, reliability function or hazard function of the gamma distribution under censoring, the large sample procedures are difficult to obtain. So we deal with this distribution here for only complete samples.

In chapter 4, we develop several test statistics for testing homogeneity of scale parameters of $(L \geq 2)$ exponential distributions under failure censoring. We then compare these statistics in terms of size and power using Monte Carlo simulations.

In chapter 5, we develop various procedures for testing homogeneity of L($\geq$ 2) Weibull or extreme value populations. We develop several test statistics for testing the equality of extreme value location parameters with the assumption of common scale parameter. We then conduct a simulation study to compare the performance of these statistics in terms of size and power. Testing for the assumption of common scale parameter is also considered. In this case, we develop various test procedures and compare the test statistics in terms of size and power by simulations.

In chapter 6, we derive several approximate procedures for constructing confidence intervals for the parameters of exponential distribution under time censoring. We then investigate the behaviour of these intervals in terms of average lengths of the confidence intervals and coverage probabilities by performing simulation study.

Chapter 7 develops procedures for setting confidence intervals for the extreme value distribution parameters under failure censoring. These results are extended to extreme value regression models under failure censoring. A simulation study is conducted to determine the behaviour of the confidence intervals. Confidence interval procedures for the parameters of extreme value models under time censoring are considered in chapter 8.

In chapter 9, I make an attempt to identify some gaps in the literature including my thesis and propose some plans for future study.

CHAPTER 2

SOME BASIC DEFINITIONS, CONCEPTS AND MODELS OF INTEREST

2.1. SOME DEFINITIONS

Cumulative Distribution

Let T be a continuous non-negative random variable that represents the lifetime of an item in a specified population with parameter θ, where θ may be a scalar or vector valued. Then the probability density function (pdf) is denoted by $f(t;\ \theta)$ and the cumulative distribution function (cdf) is denoted by $F(t;\ \theta)$, which is given by

$$F(t;\ \theta) = P(T \leq t) = \int_0^t f(x;\ \theta)dx.$$

Location Parameter

Let X be a continuous non- negative random variable. Suppose $X \sim F(X;\ u)$. If $Z = (X-u) \sim G(Z)$, where the distribution $G(Z)$ does not depend on u, then u is a location parameter; that is $F(X;\ u) = G(X-u)$.

Scale Parameter

Suppose $X \sim F(X;\ b)$. If $Z = X/b \sim G(Z)$, where the distribution $G(Z)$ does not depend on b, then b is a scale parameter; that is $F(X;\ b) = G(X/b)$.

Location- Scale Parameters

Suppose $X \sim F(X;\ u, b)$. If $Z = (X-u)/b \sim G(Z)$, where the distribution $G(Z)$ does not depend on the parameters u and b then u and b are called location- scale parameters;

that is

$F(X; u, b) = G((X-u)/b)$.

2.2 MAXIMUM LIKELIHOOD ESTIMATION (MLE)

Suppose $\theta = (\theta_1,\ldots,\theta_p)' \in \Omega$, the parameter space and $L(\theta) = f(X_1,\ldots,X_n; \theta)$ is the joint probability density function of n random variables $X_1,\ldots,X_n$. The maximum likelihood estimates of the parameters $\theta_1,\ldots,\theta_p$ are obtained by solving

$$\frac{\partial L(\theta)}{\partial \theta_i}=0, \quad i=1,\ldots,p. \tag{2.2.1}$$

Since $L(\theta)$ and logarithm of $L(\theta)$ have their maxima at the same value of θ, it is often more convenient to work with logarithm of $L(\theta)$.

The solutions of the system of likelihood equations in (2.2.1) are not always available explicitly. In this situation, solutions of the above system of likelihood equations or maximization of likelihood function can be obtained iteratively. For this problem, various approaches have been suggested by several authors. For example: Pike(1966) used the Hooke-Jeens derivative free search procedure to obtain the maximum likelihood estimators of the Weibull parameters. Jenkinson(1969) described an iterative procedure for the parameters of the generalized extreme value distribution. Johnson and Kotz(1970) have given an iterative method for the maximum likelihood estimators for the parameters of extreme value distribution. Archer(1980) proposed a hybrid technique for solving this problem. But one of the main well- known approaches to solving these problems, used by numerous researchers, is the Newton-Raphson method. However, in this thesis, the system of non- linear equations has been solved iteratively by using the appropriate IMSL subroutines, such as the DZBREN, DZREAL and the DNEQNF.

2.3 FISHER INFORMATION MATRIX

Elements of the Fisher information matrix are minus the expected values of the second order mixed partial derivatives of the log likelihood function with respect to the parameters. Suppose $l(X, \theta, \phi)$ is the log likelihood function and $\theta = (\theta_1,...,\theta_p)'$ and $\phi = (\phi_1,...,\phi_p)'$ then the Fisher information matrix I is given by

$$I = \begin{bmatrix} -E\left(\frac{\partial^2 l}{\partial\theta\partial\theta'}\right) & -E\left(\frac{\partial^2 l}{\partial\theta\partial\phi'}\right) \\ -E\left(\frac{\partial^2 l}{\partial\phi\partial\theta'}\right) & -E\left(\frac{\partial^2 l}{\partial\phi\partial\phi'}\right) \end{bmatrix}$$

2.4 VARIANCE- COVARIANCE MATRIX

Suppose $\hat{\theta}$ and $\hat{\phi}$ are the maximum likelihood estimates of θ and ϕ respectively. Then the asymptotic variance covariance matrix of $(\hat{\theta}, \hat{\phi})$ is obtained by inverting the Fisher information matrix.

2.5 ROOT-N CONSISTENT ESTIMATOR

A sequence of estimates $\{\hat{\theta}_n\}$, n = 1,2,..., is said to be root-n consistent estimate for the parameter θ if the quantity $\sqrt{n}\ |\hat{\theta}_n-\theta|$ remains bounded in probability as n tends to infinity.

Corollary: Let $\{\hat{\theta}_n\}$, n = 1,2,..., be a sequence of estimates of θ such that variance of $(\hat{\theta}_n)$ tends to zero when n tends to infinity, then the sequence of estimates $\{\hat{\theta}_n\}$, n = 1,2,..., is root-n consistent.

Proof: Following Chebychev's inequality, for a given $\in > 0$,

$$P(\sqrt{n}\ |\ \hat{\theta}_n-\theta\ | < \in) \geq 1 - n\ Var(\hat{\theta}_n) / \in^2$$

From the definitions of 2.3 and 2.4, if $\hat{\theta}_n$ is the MLE of θ, then by the asymptotic properties of MLE, $Var(\hat{\theta}_n)$ tends to zero as n tends to infinity. That is, $Var(\hat{\theta}_n)$ is $O(n^{-1})$ (Kendall and Stuart, Vol. 2, p. 51). Thus, MLE is root-n consistent.

2.6 PIVOTAL QUANTITY

Suppose $X_1,\ldots, X_n$ is a sample of size n from a location- scale family of the form

$$f(X;u,b)=\frac{1}{b}\ g\left(\frac{X-u}{b}\right)$$

where u and b are the location - scale parameters. Let $\hat{u}$ and $\hat{b}$ be the maximum likelihood estimators of the parameters u and b respectively. Then the estimators $\hat{u}$ and $\hat{b}$ have the property that the pivotal quantities $(\hat{u}-u)/\hat{b}$ and $\hat{b}/b$ are distribution free of the parameters u and b.

2.7 ORDER STATISTICS

Suppose that a random variable X has a pdf f(X) and cdf F(X). The random sample $X_1,\ldots, X_n$, of size n is rearranged in the order of magnitude and is denoted by $X_{(1)} \leq \ldots \leq X_{(n)}$. Then the variables $X_{(1)},\ldots, X_{(n)}$ are called the order statistics of the sample.

Result 1:

The joint pdf of $X_{(1)},\ldots, X_{(r)}$, $(r \leq n)$, is

$$f(x_{(1)},\ldots,x_{(r)})=\frac{n!}{(n-r)!}\left(\prod_{i=1}^{r} f(x_{(i)})\right)\left[\ 1-F(x_{(r)})\right]^{(n-r)}.$$

Result 2:

The pdf of $X_{(i)}$, $(1 \leq i \leq n)$, is

$$f(x_{(i)})=\frac{n!}{(i-1)!(n-i)!}\, f(x_{(i)})\,\left[F(x_{(i)}\right]^{(i-1)}\left[\ 1-F(x_{(i)}\right]^{(n-i)}.$$

Result 3:

The joint pdf of $X_{(i)}$ and $X_{(j)}$, $(1 \leq i < j \leq n)$, is

$$f(x_{(i)},x_{(j)})=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\, f[x_{(i)}]\,\left[f(x_{(j)})\right]\left[F(x_{(j)})\right]^{(i-1)}$$
$$\left[1-F(x_{(j)})\right]^{(n-j)}\ \left[F(x_{(j)})-\ F(x_{i})\right]^{(j-i-1)}.$$

The derivations of the above results can be found in several books (see David 1970; Sarhan and Greenberg, 1962; etc.).

2.8 TYPES OF LIKELIHOODS

Suppose $\delta = (\theta,\phi)' \in \Omega$, the parameter space, where $\theta = (\theta_1,\ldots,\theta_p)'$ are of interest and $\phi = (\phi_1,\ldots,\phi_p)'$ are treated as nuisance parameters. Let $X = X_1,\ldots, X_n$ be a random sample of size n. Then the likelihood function is given by $L(X;\ \theta,\ \phi) = f(X;\ \theta,\ \phi)$. To eliminate the nuisance parameter from the likelihood function $L(X;\ \theta,\ \phi)$, various methods have been proposed in the statistical literature (see Kalbfleisch and Sprott, 1970; Cox and

Reid, 1987).

2.8.1 Integrated Likelihood

If the prior density of ϕ, $g(\phi, \theta)$, is known then the integrated likelihood function denoted by $L_I(\theta)$ is given by

$$L_I(\theta) = \int_\phi f(X; \theta|\phi)\ g(\phi,\theta)\ d\phi.$$

2.8.2 Marginal Likelihood

This method depends on factoring the likelihood function into two parts. One contains the parameter θ only, which is of interest, and the other has a joint likelihood for θ and ϕ, where ϕ is treated as nuisance parameter. Let $a_1,\ldots, a_r$ be jointly ancillary for ϕ, for a given θ. Suppose there exists a non singular transformation $(X_1,\ldots, X_n)$---> $a_1,\ldots,a_r,Y_1,\ldots, Y_{(n-r)}$ such that

$$f(x; \theta, \phi)dx_1,\ldots,dx_n=[f(a_1,\ldots, a_r)da_1,\ldots,da_r]$$

$$[g(y_1,\ldots y_{(n-r)}; \theta, \phi \mid a_1,\ldots,a_r)dy_1,\ldots,dy_{(n-r)}].$$

If the conditional density g contains no information on θ when ϕ is unknown, the marginal likelihood $L_m(\theta)$ is defined as a function of θ, which is given by the joint density of $(a_1,\ldots, a_r)$; that is

$L_m(\theta) = f(a_1,\ldots,a_r)\ da_1,\ldots, da_r.$

2.8.3 Conditional Likelihood

Assume that for a given θ, there exists a minimal sufficient statistic T, with pdf

$g(T;\ \theta,\phi)$, for the estimation of ϕ such that

$f(X;\ \theta,\ \phi) = h(X \mid T;\ \theta)\ \ g(T;\ \theta,\ \phi)$

Then $h(X \mid T;\ \theta)$, which has a distribution involving θ only, is called conditional likelihood of θ; that is

$L_c(\theta) = h(X_1, \ldots, X_n \mid T;\ \theta)$.

The main advantage of these methods is that the resulting likelihoods $L_1(\theta)$, $L_m(\theta)$ or $L_c(\theta)$ depend only on the parameter of interest. However, the elimination of some parameters may result in loss of information relative to that contained in the full likelihood $L(\theta,\ \phi)$.

2.9 RELIABILITY FUNCTION (SURVIVOR FUNCTION)

Let T be a continuous non-negative random variable representing the lifetime of an item from a specified population. Further, let $F(t;\ \theta)$ and $f(t;\ \theta)$ be the cdf and pdf of T respectively, where θ is an unknown parameter. Then the reliability function $R(t;\ \theta)$ is defined by the probability of an item surviving at least until time t; that is

$$R(t;\ \theta) = P(T \geq t) = \int_t^\infty f(x;\ \theta)dx.$$

In life sciences, the reliability function $R(t;\ \theta)$ is referred to as the survivor function and denoted by $S(t;\ \theta)$. It is obvious that $R(t;\ \theta)$ is a monotone decreasing continuous function with $R(0,\ \theta) = 1$ and

$$R(\infty,\ \theta) = \lim_{t \to \infty} R(t;\ \theta) = 0.$$

2.10 HAZARD FUNCTION

The instantaneous failure rate is often referred to as the hazard function, $h(t;\ \theta)$, since it describes the way in which the instantaneous probability of a failure changes with time. The hazard function $h(t;\ \theta)$ is defined as

$$h(t;\ \theta) = \lim_{h \to 0} \frac{P[t \le T \le t+h \mid T \ge t]}{h}$$

which can be written as

$$h(t;\ \theta) = \lim_{h \to 0} \frac{P[t \le T \le t+h,\ T \ge t]}{h\ P[T \ge t]} = \frac{f(t;\ \theta)}{R(t;\ \theta)}.$$

In the actuarial field, the hazard rate $h(t;\ \theta)$ is referred to as the force of mortality.

2.11. RELATIONSHIP AMONG f, F, R AND h

$$f(t;\ \theta) = F'(t;\ \theta) = -R'(t;\ \theta) = -\frac{d}{dt} R(t;\ \theta).$$

Since $h(t;\ \theta) = f(t;\ \theta)/R(t;\ \theta)$, we obtain

$$h(t;\ \theta) = -\frac{d}{dt}\log R(t;\ \theta),$$

Thus,

$$R(t;\ \theta) = \exp\left(-\int_0^t h(x;\ \theta)dx \right).$$

2.12 TYPES OF CENSORING

2.12.1 Type II (Failure) Censoring

Suppose a random sample contains n units and is placed on a life test. The first r failure times are observed. Denote the ordered failure times by $T_1 \leq ... \leq T_r$. When the unfailed units exceed a time $L = T_r$, the sample is said to be type II censored or failure censored.

Now $T_1 \leq ... \leq T_r$ out of a random sample of size n are i.i.d and have a continuous distribution with pdf $f(t; \theta)$ and reliability function $R(t; \theta)$. From section 2.7, the joint pdf of $T_1, ..., T_r$, which is the likelihood function, is given by

$$L(\theta) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{r} f(t_i; \theta)\right)\left(R(t_r\ ;\ \theta)\right)^{(n-r)}.$$

2.12.2 Type I (Time) Censoring

Let n items be placed on a life test. The experiment is terminated after a predefined time L_i for the ith item; that is, the lifetime T_i of the ith item, is observed only if $T_i \leq L_i$, $i = 1,...,n$. In this situation the data are said to be type I censored or time censored.

Suppose T_i's are assumed to be i.i.d with pdf $f(t; \theta)$ and reliability function $R(t; \theta)$. For convenience, we define

$$\delta_i = \begin{cases} 1\ , & T_i \leq L_i \\ 0\ , & T_i > L_i\ , \end{cases}$$

where δ_i indicates whether the lifetime T_i is uncensored or censored; that is, the time

$$t_i = \underset{1 \le i \le n}{Min} \ (T_i, L_i) = \begin{cases} T_i, & T_i \le L_i \\ L_i, & T_i > L_i \end{cases}.$$

Then the likelihood function is defined as

$$L(\theta) = \prod_{i=1}^{n} \left[(f(t_i;\ \theta))^{\delta_i} \ (R(L_i;\ \theta))^{(1-\delta_i)} \right].$$

When $L_i = L$, $i = 1, \ldots, n$, the above sample is referred to as being singly (time) censored. There are other types of censoring (see Lawless, 1982) which we do not deal with in this thesis.

2.13 MODELS OF INTEREST

2.13.1 Gamma Distribution

The Gamma distribution is available as a model in meteorology, life testing and reliability studies. The two parameter gamma distribution of a random variable T has a pdf of the form

$$f(t;\ \lambda,\ k) = \frac{t^{(k-1)}\ e^{(-t\ /\ \lambda)}}{\lambda^k\ \Gamma(k)}, \qquad t,\ k,\ \lambda \ge 0 \tag{2.13.1}$$

where λ is the scale parameter and k is the shape parameter. Reliability and hazard functions of the gamma distribution involve the incomplete gamma function which is given by

$$I(k;\ x) = \frac{1}{\Gamma(k)} \int_0^x y^{(k-1)}\ e^{(-y)}\ dy.$$

Now, the reliability function and the hazard function can be written as

$R(t;\lambda,k) = 1 - I(k, t/\lambda)$

$h(t;\lambda,k) = f(t,\lambda,k) / R(t;\lambda,k).$

When $k > 1$, it can be shown that the hazard function is a monotone increasing function with $h(0;\lambda,k) = 0$ and

$$h(\infty;\lambda,k) = \lim_{t \to \infty} h(t;\lambda,k) = \frac{1}{\lambda}.$$

When $\lambda = 1$, the distribution (2.13.1) reduces to the one parameter gamma distribution with pdf

$$f(t;k) = \frac{t^{(k-1)}\, e^{-t}}{\Gamma(k)}, \qquad t, k \geq 0.$$

When $k = 1$, the distribution (2.13.1) becomes the well-known one parameter exponential distribution with pdf

$$f(t;\lambda) = \frac{1}{\lambda}\, e^{(-t/\lambda)}, \qquad t,\lambda \geq 0.$$

2.13.2 Exponential Distribution

This distribution is widely employed as a model in areas such as studies on the lifetimes of manufactured items and studies involving remission times in bio-medical sciences. The pdf of a random variable T having a one parameter exponential distribution is given by

$$f(t;\, \theta) = \frac{1}{\theta}\, e^{(-t\,/\,\theta)}, \qquad t \geq 0. \tag{2.13.2}$$

The pdf of T having a two parameter exponential distribution is given by

$$f(t;\, \mu,\, \theta) = \frac{1}{\theta} \exp\left[-\left(\frac{t-\mu}{\theta}\right)\right], \qquad t \geq 0 \tag{2.13.3}$$

where μ and θ are location- scale parameters. The parameter μ is also referred to as threshold parameter. In lifetime analysis, it is often assumed that $\mu \geq 0$. In life sciences, the model (2.13.3) is often applicable in situations where it is assumed that death cannot occur before some predefined time μ. The reliability (survivor) function and the hazard function are given by

$R(t;\, \mu,\, \theta) = \exp\,[-(t-\mu)\,/\,\theta]$ and $h(t;\, \mu,\, \theta) = 1\,/\,\theta$.

Since the hazard function is constant, it is a useful model for lifetime data where used items are to be considered as good as new ones.

2.13.3 Weibull Distribution

This is the most popular lifetime distribution in practice, particularly in the field of engineering, manufacturing, bio- medical science and many other studies. The pdf of a random variable T having a two parameter Weibull distribution is given as

$$f(t;\, \alpha,\, \beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{(\beta-1)} \exp\left(-\frac{t}{\alpha}\right)^{\beta}, \qquad t \geq 0;\ \beta,\, \alpha > 0 \tag{2.13.4}$$

where the parameters α and β are, respectively, the scale and shape parameters. The reliability function $R(t;\, \alpha,\, \beta)$ and the hazard function $h(t;\, \alpha,\, \beta)$ can be written as

$$R(t;\ \alpha,\ \beta) = \exp\left(-\frac{t}{\alpha}\right)^{\beta}, \qquad t \geq 0$$

$$h(t;\ \alpha,\ \beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{(\beta-1)}, \qquad t \geq 0.$$

When $\beta = 1$, the distribution (2.13.4) reduces to

$$f(t;\ \alpha) = \frac{1}{\alpha}\exp\left(-\frac{t}{\alpha}\right), \qquad t \geq 0,$$

which is the well-known one parameter exponential distribution with parameter α. It can be shown that the Weibull hazard function is monotone increasing if $\beta > 1$, decreasing if $\beta < 1$ and constant for $\beta = 1$.

2.13.4 Extreme Value Distribution

This distribution is sometimes referred to as the Gumbel distribution (Gumbel, 1958). There are three types of extreme value distributions with cdf for a random variable X as follows (Johnson and Kotz, 1970):

$$\textit{Type I}: \quad F(x;\ u,\ b) = \exp\left[-\exp\left(\frac{x-u}{b}\right)\right] \qquad -\infty < x,u < \infty;\ b > 0$$

$$\textit{Type II}: \quad F(x;\ u,\ b) = \begin{cases} \exp\left[-\left(\dfrac{x-u}{b}\right)^{-\delta}\right], & x \geq u \\ 0, & \textit{otherwise} \end{cases}$$

and

$$Type\ III: \quad F(x;\ u,\ b) = \begin{cases} \exp\left[-\left(\frac{x-u}{b}\right)^{\delta}\right], & x \le u. \\ 1 \quad , & otherwise \end{cases}$$

Type I is the smallest extreme value distribution and the one most often used as a model in the analysis of lifetime data. It has received extensive attention from many authors, including Lawless (1982); Nelson (1982); McCool (1979, 1980) and Escobar and Meeker (1986, 1991). In this thesis we deal with this type (Type I) of extreme value distributions.

The pdf of a random variable X from the smallest extreme value distribution is given by

$$f(x;\ u,\ b) = \frac{1}{b}\exp\left[\left(\frac{x-u}{b}\right) - \exp\left(\frac{x-u}{b}\right)\right] \tag{2.13.5}$$

where $-\infty < u < \infty$ and $b > 0$ are the location- scale parameters. This distribution is directly related to the Weibull distribution given by (2.13.4). If T has a Weibull distribution with pdf (2.13.4), it can be easily seen that $X = \log T$ has an extreme value distribution having pdf (2.13.5) with $u = \log \alpha$ and $b = \beta^{-1}$. In the light of the relationship, the extreme value distribution is sometimes referred to as log Weibull distribution. In analysing Weibull data, it is more convenient to work with log Weibull lifetimes. The transformed (log Weibull) sample is treated as one from an extreme value distribution. The reliability function R(x; u, b) and the hazard function h(x; u, b) can be given as

$$R(x;\ u,b) = \exp\left[-\exp\left(\frac{x-u}{b}\right)\right],$$

$$h(x;\ u,b) = \frac{1}{b}\exp\left(\frac{x-u}{b}\right)$$

2.14 DISTRIBUTIONAL PROPERTIES

Suppose $X_1 \leq ... \leq X_r$ are the first r ordered observations in a sample of size n from a one parameter exponential population given by (2.13.2).

Define $\quad Y_1 = n\,X_1$

$\qquad Y_i = (n-i+1)\,(X_i-X_{i-1}), \qquad i = 2,..., r.$

Then

(1) $Y_1,..., Y_n$ are independent and identically distributed with pdf (2.13.2).

(2) Let $X_s = \sum_{j=1}^{s} \frac{Z_j}{(n-j+1)}$, where $Z_j \sim \exp(\theta = 1), j = 1,..., r$ are independent.

Then

$$E(X_s) = \sum_{j=1}^{s} \frac{1}{(n-j+1)},$$

$$Var(X_s) = \sum_{j=1}^{s} \frac{1}{(n-j+1)^2}.$$

(3) If $\hat{\mu}$ and $\hat{\theta}$ are the MLEs of μ and θ respectively, then $\hat{\mu}$ and $\hat{\theta}$ are independent, and $2n(\hat{\mu}-\mu)/\theta \sim \chi^2(2)$ and $2r\hat{\theta}/\theta \sim \chi^2(2(r-1))$, where $\hat{\mu} = X_1$ and

$$\hat{\theta} = \frac{1}{r}\left[\sum_{i=1}^{r} (X_i - X_1) + (n-r)(X_r - X_1) \right].$$

(4) Let X be a random variable having the pdf (2.13.5). Then Z = (X-u)/b has standard extreme value distribution. If Z_j is the jth ordered statistic in a sample of size n, then by 2.7,

$$(a) \quad E\,(Z_j) = C_j \int_{-\infty}^{\infty} z\, e^{z}\, e^{-(n-j+1)\, e^{z}} \left(1 - e^{-e^{z}} \right)^{j-1} dz$$

$$= C_j \sum_{s=1}^{j-1} (-1)^{s-1} \binom{j-1}{s-1} \left(\frac{\gamma + \log(n-j+s)}{(n-j+s)} \right),$$

$$(b) \quad E\,(e^{Z_j}) = C_j \int_{-\infty}^{\infty} e^{2z}\, e^{-(n-j+1)\, e^{z}} \left(1 - e^{-e^{z}} \right)^{j-1} dz$$

$$= C_j \sum_{s=1}^{j-1} (-1)^{s-1} \binom{j-1}{s-1} \frac{1}{(n-j+1)^2},$$

$$(c) \quad E\,(Z_j) = C_j \int_{-\infty}^{\infty} z\, e^{2z}\, e^{-(n-j+1)\, e^{z}} \left(1 - e^{-e^{z}} \right)^{j-1} dz$$

$$= C_j \sum_{s=1}^{j-1} (-1)^{s-1} \binom{j-1}{s-1} \left(\frac{1 - \gamma - \log(n-j+s)}{(n-j+s)^2} \right),$$

(d) $E\,(Z_j^2) = C_j \int_{-\infty}^{\infty} z^2\, e^{2z}\, e^{-(n-j+1)\,e^z}\, (\,1-e^{-e^z}\,)^{j-1}\, dz$

$$= C_j \sum_{s=1}^{j-1} (-1)^{s-1} \binom{j-1}{s-1} \left(\frac{\pi^2/6 + (1 - \gamma - \log(n-j+s))^2 - 1}{(n-j+s)^2} \right),$$

where $C_j = n!/(\,(j-1)!\ (n-j)!\,)$. The expressions for the expected values in (4) are mathematically messy, so we provide their approximations in (5).

(5) Now, $V = \exp(Z)$ has standard exponential distribution. If V_j is the jth ordered statistic in a sample of size n from the standard exponential distribution, then by 2.14 (2),

$$E\,(V_j) = t_j = \sum_{s=1}^{j} \frac{1}{(n-s+1)}$$

and

$$Var(V_j) = d_j = \sum_{s=1}^{j} \frac{1}{(n-s+1)^2}$$

Using the Taylor series expansion of V_j about its mean t_j, and retaining terms up to the second order, we have

$$Z_j = \log V_j = \log\,(t_j+V_j-t_j)$$
$$\approx \log t_j + (V_j-t_j)/t_j - [(V_j-t_j)/t_j]^2/2.$$

(a) $E(Z_j) = \log t_j - d_j/\,2t_j^2$.

$$Z_j\exp(Z_j) = V_j \log V_j = (t_j+V_j-t_j)\log\,(t_j+V_j-t_j)$$
$$\approx t_j\,[1 + (V_j-t_j)/t_j]\,\log\,[\,t_j\,(1 + (V_j-t_j)/t_j)\,]$$

$$\approx t_j \log t_j + (V_j - t_j)(1 + \log t_j) + (V_j - t_j)^2/2t_j.$$

(b) $E[Z_j \exp(Z_j)] \approx t_j \log t_j + d_j/2t_j.$

$$Z_j^2 \exp(Z_j) = V_j (\log V_j)^2 = (t_j + V_j - t_j) [\log (t_j + V_j - t_j)]^2$$
$$\approx t_j (\log t_j)^2 + [(1 + \log t_j)/t_j] (V_j - t_j)^2$$
$$+ 2 (V_j - t_j) \log t_j.$$

(c) $E[Z_j^2 \exp(Z_j)] \approx t_j (\log t_j)^2 + d_j (1 + \log t_j)/t_j.$

$$Z_j^3 \exp(Z_j) = V_j (\log V_j)^3 = (t_j + V_j - t_j) [\log(t_j + V_j - t_j)]^3$$
$$\approx t_j (\log t_j)^3 + [3t_j \log t_j + 1.5\, t_j (\log t_j)^2]$$
$$[(V_j - t_j)/t_j]^2.$$

(d) $E[Z_j^3 \exp(Z_j)] \approx t_j (\log t_j)^3 + 1.5 \log t_j (2 + \log t_j) d_j/t_j.$

$$Z_j^4 \exp(Z_j) = V_j (\log t_j)^4 = (t_j + V_j - t_j) [\log (t_j + V_j - t_j)]^4$$
$$\approx t_j (\log t_j)^4 + 2 (\log t_j)^2 (3t_j + t_j \log t_j)$$
$$[(V_j - t_j)/t_j]^2.$$

(e) $E[Z_j^4 \exp(Z_j)] \approx t_j (\log t_j)^4 + 2 d_j (3 + \log t_j) (\log t_j)^2 / t_j.$

2.15 LARGE SAMPLE TEST PROCEDURES FOR COMPOSITE HYPOTHESES

Suppose $X = (X_1, \ldots, X_n)'$ is a random sample of size n taken from a particular distribution with pdf $f(X; \delta)$, where $\delta = (\theta, \phi)' = (\theta_1, \ldots, \theta_p, \phi_1, \ldots, \phi_q)'$ is a (p+q)-component vector. Then the sample likelihood can be given as $L(X_1, \ldots, X_n; \delta)$. It is of interest to test the null hypothesis H_0: $\theta = \theta_0 = (\theta_{10},\ldots,\theta_{p0})'$ and $\phi = (\phi_1,\ldots,\phi_q)'$ treated as nuisance parameters.

2.15.1 Likelihood Ratio Test

The likelihood ratio for testing H_0 is defined as

$$\Lambda = \frac{L(X_1,\ldots,X_n;\ \theta_0,\ \hat{\phi})}{L(X_1,\ldots,X_n;\ \bar{\theta},\ \bar{\phi})},$$

where $\hat{\phi}$ is the restricted MLE of ϕ and $\bar{\delta} = (\bar{\theta},\bar{\phi})'$ is the unrestricted MLE of $\delta = (\theta,\phi)'$. Denote the restricted and unrestricted maximum value of the likelihood function by L_0 and L_1 respectively. Since it is more convenient to work with the maximum log likelihoods we denote $l_0 = \log L_0$ and $l_1 = \log L_1$. Then the log likelihood ratio statistic LR, which is equivalent to the statistic Λ, is given by

$$LR = -2 \log \Lambda = 2\,(l_1 - l_0). \tag{2.15.1}$$

Under the null hypothesis H_0, distribution of LR is approximately chi-square with p degrees of freedom. It is well-known that this test is versatile and applies to most statistical distributions and to most types of data.

2.15.2 C(α) Test

Define the partial derivatives evaluated at $\theta = \theta_0 = (\theta_{10},\ldots,\theta_{p0})'$

$$\psi = \left.\frac{\partial l}{\partial \theta}\right|_{\theta=\theta_0} = \left.\left(\frac{\partial l}{\partial \theta_1},\ldots,\frac{\partial l}{\partial \theta_p}\right)'\right|_{\theta=\theta_0},$$

and

$$\eta = \left.\frac{\partial l}{\partial \phi}\right|_{\theta=\theta_0} = \left.\left(\frac{\partial l}{\partial \phi_1}, \ldots, \frac{\partial l}{\partial \phi_q}\right)'\right|_{\theta = \theta_0}.$$

It is known that under mild regularity conditions, $(\psi, \eta)'$ follows a multivariate normal distribution with mean vector 0 and variance covariance matrix I^{-1}, where

$$I = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$$

is the Fisher information matrix with elements

$$I_{11} = E\left(-\left.\frac{\partial^2 l}{\partial\theta\partial\theta'}\right|_{\theta=\theta_0}\right),$$

$$I_{12} = E\left(-\left.\frac{\partial^2 l}{\partial\theta\partial\phi'}\right|_{\theta=\theta_0}\right),$$

and

$$I_{22} = E\left(-\left.\frac{\partial^2 l}{\partial\phi\partial\phi'}\right|_{\theta=\theta_0}\right).$$

Following Neyman (1959), the $C(\alpha)$ test statistic is based on $S = (S_1,\ldots,S_p)' = \psi - B\eta$, where B is the partial regression coefficient matrix obtained by regressing $\partial l/\partial\theta$ on $\partial l/\partial\phi$. From Bartlett (1953), $B = I_{12} I^{-1}_{22}$ and the variance covariance matrix of S is $I_{11.2}$, where $I_{11.2} = I_{11} - I_{12} I^{-1}_{22} I_{21}$. Thus $S \sim MN(0,I_{11.2})$ and

$$S' I_{11.2} S \sim \chi^2_{(p)}, \tag{2.15.2}$$

where MN denotes multivariate normal.

Notice that the above expression depends on the nuisance parameter vector $\phi = (\phi_1,...,\phi_q)'$, which makes the statistic inappropriate to use for testing the null hypothesis. Moran(1970) suggested that the nuisance parameters in (2.15.2) may be replaced by their root-n consistent estimators. Let $\bar{\phi} = (\bar{\phi}_1,...,\bar{\phi}_q)'$ be a random vector of some root-n consistent estimators of the parameter vector $\phi = (\phi_1,...,\phi_q)'$, obtained from the available data. Then, replacing ϕ by $\bar{\phi}$ and θ by θ_0, and following Neyman(1959), the C(α) test statistic is defined for testing H_0: $\theta = \theta_0$, as $\chi^2_c = \bar{S}' \, I^{-1}_{11.2} \, \bar{S}$ which is asymptotically distributed as chi square with p degrees of freedom.

Note that when we replace the nuisance parameters ϕ by their MLEs $\hat{\phi}$, the score function S_i reduces to ψ_i, i = 1,...,(L-1). Then, the C(α) statistic reduces to $\hat{\psi}' \, I^{-1}_{11.2} \, \hat{\psi}$, in which situation the procedure is referred to as Score test (Rao, 1947).

The score test is asymptotically equivalent to the likelihood ratio test and tests using the maximum likelihood estimators, for example Wald tests (Moran, 1970; Cox and Hinkley, 1974). The chief advantage of the C(α) class of tests is that it maintains, at least approximately, a pre-assigned level of significance, say α (Bartoo and Puri, 1967), it is locally asymptotically most powerful (Moran, 1970) and often produces a statistic which is simple to calculate. As homogeneity tests, the C(α) class of tests have been widely used (see Neyman and Scott, 1966; Moran, 1973; Tarone, 1979; Tarone, 1985; Barnwal and Paul, 1988; Paul, 1989; etc.).

2.16 INTERVAL ESTIMATION PROCEDURES

Let $f(X; \theta, \phi)$ be a density of a random variable X indexed by θ and ϕ, where θ is the parameter of interest and $\phi = (\phi_1, ..., \phi_p)'$ is a vector of p nuisance parameters. Given the sample $X_1, ..., X_n$, denote the log likelihood by $l(\theta, \phi)$.

2.16.1 Procedure Based on the Asymptotic Properties of MLE

Denote the MLEs of the parameters θ and $\phi = (\phi_1, ..., \phi_q)'$ by $\hat{\theta}$ and $\hat{\phi} = (\hat{\phi}_1, ..., \hat{\phi}_q)'$ respectively. The asymptotic $100(1-\alpha)\%$ confidence interval for θ is given by

$$\hat{\theta} - \zeta \sqrt{var(\hat{\theta})} < \theta < \hat{\theta} + \zeta \sqrt{var(\hat{\theta})}, \tag{2.16.1}$$

where ζ is an appropriate quantile of a standard normal random variable. The quantity $var(\hat{\theta})$ can be obtained from the Fisher information matrix of $(\hat{\theta}, \hat{\phi})$ as discussed in section 2.3. It is to be noted that $100(1-\alpha)\%$ confidence interval for θ can also be used to test a null hypothesis about θ at the $100\alpha\%$ of significance level. Thus, if a hypothetical value θ_0 of θ lies outside the confidence interval, then the null hypothesis $H_0: \theta = \theta_0$ is rejected at the $100\alpha\%$ significance level, in favour of an alternative hypothesis $H_1: \theta \neq \theta_0$; otherwise H_0 is not rejected.

2.16.2 Procedure Based on Likelihood Ratio

Denote the unconstrained maximum log likelihood by $l(\hat{\theta}, \hat{\phi})$, and the constrained maximum log likelihood by $l(\theta, \bar{\phi})$, where $\bar{\phi} = (\bar{\phi}_1, ..., \bar{\phi}_p)'$ are the values of $\phi = (\phi_1, ..., \phi_p)'$ that maximize the log likelihood function $l(\theta, \phi)$ for a given value of θ. Then the likelihood ratio statistic $LR = 2\,[l(\hat{\theta}, \hat{\phi}) - l(\theta, \bar{\phi})]$ has a distribution which is approximately chi-square with one degree of freedom. Thus, the θ values that satisfy

$$LR = 2\,[l(\hat{\theta}, \hat{\phi}) - l(\theta, \bar{\phi})] = \chi^2_{(1-\alpha)}(1) \tag{2.16.2}$$

are the approximate 100(1-α)% confidence limits for θ, where $\chi^2_{(1-\alpha)}(1)$ is the (1-α)th quantile of a chi squared density with one degree of freedom.

2.16.3 Procedure Based on Likelihood Score Corrected for Bias and Skewness

Define

$$I_{\theta\theta} = E\left(-\frac{\partial^2 l}{\partial\theta^2}\right), \quad I_{\theta\phi} = E\left(-\frac{\partial^2 l}{\partial\theta\partial\phi}\right) \ \ and \ \ I_{\phi\phi} = E\left(-\frac{\partial^2 l}{\partial\phi\partial\phi'}\right),$$

where $I_{\theta\phi}$ is of order 1xp and $I_{\phi\phi}$ is of order p x p, and $I_{\theta\theta.\phi} = I_{\theta\theta} - I_{\theta\phi}\, I^{-1}_{\phi\phi}\, I_{\phi\theta}$. Bartlett (1953) proposed a procedure based on the likelihood score in the presence of nuisance parameters for constructing confidence interval for a single parameter. Now, we define

$$T_\theta = \frac{\partial l}{\partial\theta} - I_{\theta\phi}\, I_{\phi\phi}^{-1}\, \frac{\partial l}{\partial\phi},$$

where

$$\frac{\partial l}{\partial\phi} = \left(\frac{\partial l}{\partial\phi_1}, \ldots, \frac{\partial l}{\partial\phi_p}\right)'.$$

For convenience, we define $f = (f_1,\ldots,f_p)' = I_{\theta\phi}\, I^{-1}_{\phi\phi}$. Then T_θ can be written as

$$T_\theta = \frac{\partial l}{\partial\theta} - f \cdot \frac{\partial l}{\partial\phi}.$$

Bartlett (1953) showed that T_θ is asymptotically distributed normal with mean zero and variance $I_{\theta\theta.\phi}$. Thus an approximate 100(1-α)% confidence interval for θ can be obtained by solving

$$\frac{T_\theta}{\sqrt{I_{\theta\theta.\phi}}} = \pm \zeta ,$$

where ζ is an appropriate quantile of a standard normal random variate. However, when the nuisance parameters ϕ are replaced by their maximum likelihood estimators for a given θ, the statistic T_θ has a bias of order $O(n^{-1/2})$ and is given by

$$Bias = B\,(T_\theta) = -\frac{1}{2}\, trace \left\{ I_{\phi\phi}^{-1} \left[E\left(\frac{\partial^3 l}{\partial\theta\partial\phi\partial\phi'}\right) + 2\,\frac{\partial I_{\theta\phi}}{\partial\phi} \right]\right\} + \frac{1}{2}\, trace \left[I_{\phi\phi}^{-1}\, M \right],$$

where

$$M_j = \left[E\left(\frac{\partial^3 l}{\partial\phi_j\partial\phi\partial\phi'}\right) + 2\,\frac{\partial I_{\phi\phi}}{\partial\phi_j}\right] I_{\phi\phi}^{-1}\, I_{\theta\phi}, \quad j=1,\dots,p,$$

(see Bartlett, 1955; Levin and Kong, 1990). The third cumulant of T_θ, to the order $o(n^{-3/2})$, is obtained for s, t, q = 1,..., p, as

$$K_3(\theta) = 2\,E\left(\frac{\partial^3 l}{\partial\theta^3}\right) + 3\,\frac{\partial I_{\theta\theta}}{\partial\theta} - 3\sum_s f_s \left[2\,E\left(\frac{\partial^3 l}{\partial\theta^2\partial\phi_s}\right) + 2\,\frac{\partial I_{\theta\phi_s}}{\partial\theta} + \frac{\partial I_{\theta\theta}}{\partial\phi_s}\right]$$

$$+ 3\sum_s\sum_t f_s f_t \left[2\,E\left(\frac{\partial^3 l}{\partial\theta\partial\phi_s\partial\phi_t}\right) + \frac{\partial I_{\phi_s\phi_t}}{\partial\theta} + \frac{\partial I_{\theta\phi_t}}{\partial\phi_s} + \frac{\partial I_{\theta\phi_s}}{\partial\phi_t}\right]$$

$$- \sum_s \sum_t \sum_q f_s f_t f_q \left[2\, E\left(\frac{\partial^3 l}{\partial\phi_s \partial\phi_t \partial\phi_q} \right) + \frac{\partial I_{\phi_t\phi_q}}{\partial\phi_s} + \frac{\partial I_{\phi_s\phi_q}}{\partial\phi_t} + \frac{\partial I_{\phi_s\phi_t}}{\partial\phi_q} \right].$$

Now, the statistic T_θ corrected for bias and skewness is better approximated by the normal distribution and thus an approximate 100(1-α)% confidence interval for θ can be obtained by solving

$$\frac{T_\theta}{\sqrt{I_{\theta\theta.\phi}}} - \frac{B\,(T_\theta)}{\sqrt{I_{\theta\theta.\phi}}} - \frac{K_3(\theta)\ (\zeta^2-1)}{6\ (I_{\theta\theta.\phi})^{3/2}} = \pm\, \zeta\,. \qquad (2.16.3)$$

When p = 1, we deal with the confidence interval procedure for the parameter θ in the presence of a single nuisance parameter ϕ. Then, $f = I_{\theta\phi}\, I^{-1}_{\phi\phi}$,

$$T_\theta = \frac{\partial l}{\partial\theta} - f\,\frac{\partial l}{\partial\phi}\,,$$

$$B\,(T_\theta) = -\frac{1}{2} I^{-1}_{\phi\phi} \left\{ E\left(\frac{\partial^3 l}{\partial\theta\partial\phi^2} \right) + 2\,\frac{\partial I_{\theta\phi}}{\partial\phi} - f\left[E\left(\frac{\partial^3 l}{\partial\phi^3} \right) + 2\,\frac{\partial I_{\phi\phi}}{\partial\phi} \right] \right\},$$

and

$$K_3\,(\theta) = 2\,E\left(\frac{\partial^3 l}{\partial\theta^3} \right) + 3\,\frac{\partial I_{\theta\theta}}{\partial\theta} - 3\,f\left[2\,E\left(\frac{\partial^3 l}{\partial\theta^2\partial\phi} \right) + 2\,\frac{\partial I_{\theta\phi}}{\partial\theta} + \frac{\partial I_{\theta\theta}}{\partial\phi} \right]$$

$$+\ 3\,f^2\left[2\,E\left(\frac{\partial^3 l}{\partial\theta\partial\phi^2} \right) + \frac{\partial I_{\phi\phi}}{\partial\theta} + 2\,\frac{\partial I_{\theta\phi}}{\partial\phi} \right]$$

$$- f^3 \left[2\, E \left(\frac{\partial^3 l}{\partial \phi^3} \right) + 3\, \frac{\partial I_{\phi\phi}}{\partial \phi} \right].$$

Now, if ϕ is orthogonal to θ then $I_{\theta\phi} = 0$, and the above expressions for T_θ, E (T_θ) and $K_3(\theta)$ simplify as follows:

$$T_\theta = \frac{\partial l}{\partial \theta},$$

$$B\,(T_\theta) = -\frac{1}{2}\, I_{\phi\phi}^{-1} \left[E \left(\frac{\partial^3 l}{\partial \theta \partial \phi^2} \right) \right],$$

and

$$K_3(\theta) = 2\, E \left(\frac{\partial^3 l}{\partial \theta^3} \right) + 3\, \frac{\partial I_{\theta\theta}}{\partial \theta}\ .$$

2.16.4 Procedure Based on Adjusted Likelihood Ratio

Diciccio, Field and Fraser (1990) developed a confidence interval procedure for the parameters of a location-scale family of distributions, where the location may be a function of several regressor variables $X_1,\ldots, X_p$. Thus, if $p = 1$, we will deal with confidence interval procedures for the parameters of a two parameter distribution. To keep in line with the notations in this section let $\phi_1,\ldots, \phi_p$ be the regression parameters and θ be the scale parameter. Note that any of these p+1 parameters may be of interest. Now, we define

$$V_s = (\phi_s - \hat{\phi}_s)/\hat{\theta}\ ,\ s = 1,\ldots, p$$

and

$$V_{p+1} = \log (\theta/\hat{\theta}),$$

where $\hat{\phi} = (\hat{\phi}_1,\ldots, \hat{\phi}_p)'$ and $\hat{\theta}$ are the MLEs of $\phi = (\phi_1,\ldots, \phi_p)'$ and θ respectively. Let Y_i ($i = 1,\ldots,n$) be the ith value of the response variable and X_{ij} ($i = 1,\ldots,n$; $j = 1,\ldots,p$) be the ith value of the jth regressor variable. Denote the parameter free quantities $A_i = (Y_i - X_i\hat{\phi})/\hat{\theta}$, $i = 1,\ldots,n$, where $X_i\hat{\phi} = X_{i1}\hat{\phi}_1 + \ldots + X_{ip}\hat{\phi}_p$. Then for given A_i, $i = 1,\ldots,n$, the log likelihood $l(\theta,\phi)$ can be written in terms of a vector of pivotals $V = (V_1,\ldots,V_p)'$. We denote this by $l(V)$. It is obvious that the log likelihood $l(V)$ attains its maximum value $l(0)$ at $V_s = 0$, $s = 1,\ldots, p+1$.

Suppose the jth parameter is of interest. Then the associated pivotal is V_j, and the corresponding LR statistic $LR_j = 2\,[\, l(0) - l(\tilde{V}(V_j))\,]$, where $l(\tilde{V}(V_j))$ is the maximized log likelihood function for a given value of V_j. The statistic LR_j is approximately distributed as chi-square with one degree of freedom. Now we define

$$SR_j = \begin{cases} -\sqrt{LR_j} & , \quad V_j < 0 \\ \sqrt{LR_j} & , \quad V_j > 0 \end{cases} .$$

The distribution of SR_j can be approximated by the standard normal distribution, which has error of order $O(n^{-1/2})$; that is $P(V_j \le v_j) = \Phi(SR_j) + O(n^{-1/2})$,

where Φ is the distribution function of a standard random variable. Many authors including Barndorff- Nielsen (1986), Diciccio (1984, 1988) and McCullagh (1984) concluded that mean and variance adjustments to the distribution of SR_j provide better approximation to the standard normal distribution. These adjustments reduce the error to the order $O(n^{-3/2})$, and thus the marginal distribution of the pivotal V_j is given by

$$P(V_j \le v_j) = \Phi\left(\frac{SR_j - \mu_{SR_j}}{\sigma_{SR_j}}\right) + O(n^{-3/2}),$$

where μ_{SR_j} and $\sigma^2_{SR_j}$ are the mean and variance of the variable SR_j. Since exact values

of the mean and variance of SR_j are not easily obtained, Diciccio, Field and Fraser (1990) provide approximate expressions for the mean and variance of SR_j using higher order partial derivatives. They obtained the marginal tail probability for the pivotal V_j, given by

$$P(V_j \le v_j) = \Phi(SR_j) + \phi(SR_j)\left[\frac{1}{SR_j} + \frac{|I^0|^{1/2}}{l_j(\bar{V}(V_j))\ |I^*|^{1/2}}\right] + O(n^{-3/2}),$$

(2.16.4)

where ϕ is the density function of N(0,1) variable,

I^0 is the observed information matrix of order $(p+1)\mathrm{x}(p+1)$,

I^* is a sub matrix of I^0 corresponding to $V_1, \ldots, V_{j-1}, V_{j+1}, \ldots, V_{p+1})$,

$|I^0|^{1/2}$ and $|I^*|^{1/2}$ are the square root of the determinants of the matrices I^0 and I^* respectively,

and for $j = 1, \ldots, (p+1)$,

$$l_j(\tilde{V}(V_j)\ ) = \left.\frac{\partial l}{\partial V_j}\right|_{\tilde{V}(V_j)} .$$

Thus, 100(1-α)% approximate lower and upper confidence limits V^j_L and V^j_U of V_j are obtained by solving

$P(V_j \le v_j) = \alpha/2$

and

$P(V_j \le v_j\ ) = (1-\alpha/2)$

respectively. The confidence limits for the jth parameter of interest can then be obtained from V^j_L and V^j_U.

PART I

HOMOGENEITY TESTS IN MULTIPLE SAMPLES
FROM
TWO PARAMETER
GAMMA, EXPONENTIAL, WEIBULL
OR
EXTREME VALUE DISTRIBUTIONS

CHAPTER 3

TESTING HYPOTHESES IN MULTIPLE SAMPLES FROM TWO PARAMETER GAMMA DISTRIBUTION

3.1 INTRODUCTION

The gamma distribution is widely used in various hydrological, meteriological, reliability and life-testing applications. Many authors have studied this distribution. Gupta and Groll (1961) discussed the use of the gamma distribution in acceptance sampling based on life tests. Simpson (1972) studied its use in single-cloud rainfall analysis. For a general review, including numerous references to applications in diverse fields, see Johnson and Kotz (1970).

The gamma distribution has the pdf as in equation (2.13.1)

$$f(t;\lambda,k) = \frac{1}{\lambda^k\,\Gamma(k)}\, t^{k-1}\, e^{-t/\lambda}, \qquad t,\ \lambda,\ k > 0. \tag{3.1.1}$$

The mean is $\mu = k\lambda$ and the variance is $k\lambda^2 = \mu\lambda = \mu^2/k = \sigma^2\mu^2$, where $\sigma^2 = 1/k$. Depending on the value of λ , the distribution has mean = variance ($\lambda = 1$), under dispersion ($\lambda < 1$) and over dispersion ($\lambda > 1$). The analogy is with the mean-variance relationship of the Poisson, binomial and negative binomial distributions. When $k = 1$ the distribution is the well-known exponential distribution with the same mean-variance relationship as already discussed. The gamma distribution with integer value of k, called the Erlangian distribution (Cox, 1962), arises in a fairly natural way as the time to the kth event in a Poisson process. The distribution also belongs to the natural exponential family of distributions (McCullagh and Nelder, 1989). The parameter σ^2 ($= 1/k$) can be considered as the precision parameter (McCullagh and Nelder, 1989). Inference procedures for the parameters k and λ based on a single sample have been discussed by numerous authors (Lawless, 1982; Nelson, 1982; etc.). In this chapter we deal with hypothesis testing in multiple samples assumed to have come from two parameter gamma distributions.

In the area of life-testing, the limited hazard rate is $1/\lambda$. Thus in multiple samples from gamma distributions testing the homogeneity of shape parameters is equivalent to testing for homogeneity in precision and testing equal scale parameters with the assumption of common shape parameter is equivalent to testing the homogeneity of means or limiting hazard rates. It is therefore necessary to study tests for homogeneity of the scale parameters as well as shape parameters.

Shiue and Bain (1983) developed an approximate one sided test based on the ratio of the means of two samples for testing the equality of scale parameters of two gamma distributions with common shape parameter, and showed that when the unknown shape parameter is replaced by its MLE, the proposed statistic follows an approximate F distribution. Shiue, Bain and Engelhardt (1988) extended this test for testing the equality of two gamma distribution means in presence of unspecified shape parameters. Gastwirth and Mahmoud (1986) proposed a maximum efficiency robust test for testing whether two samples come from a common gamma distribution against the alternative that they differ in scale for situation with increasing hazard rate; that is the value of the shape parameter, k, is known to be > 1.

By generalizing Shiue and Bain's statistic we develop an extremal scale parameter ratio statistic(EP) for testing the homogeneity of sevaral scale parameters with a common shape parameter. For $L = 2$ and sample sizes $n_1 = n_2$ the statistic EP has a truncated F distribution shown in section 3.3.4. Otherwise, the distribution of EP is not known and a test based on this statistic has to be performed by using simulated percentage points.

In section 3.2, we describe and develop estimators of the parameters under various null and alternative hypotheses. In section 3.3, we derive and develop various test procedures such as a likelihood ratio statistic(LR), two modified likelihood ratio statistics(M and MB), a $C(\alpha)$ statistic(CL) and the extremal scale parameter ratio statistic(EP). Performance of these statistics, in terms of size and power, are studied by Monte Carlo simulation and are presented in section 3.4. All these tests have been developed assuming a common shape parameter k across the populations. However, the assumption of common shape parameter may not always be appropriate in practical context. Therefore we develop procedures for testing the assumption of common k. For this, we derive a likelihood ratio statistic(LRk) and a $C(\alpha)$ statistic(CLk) in section 3.5. The behaviour of these two test statistics, in terms of size and power, are examined by conducting a small scale simulation study and the results of the study are reported in section 3.6. Some examples are given in section 3.7. As discussed in chapter 1, the development of necessary relevant theory pertains to only complete sample situations.

3.2. ESTIMATION OF THE PARAMETERS

Consider L samples from gamma distributions, given by (3.1.1), with parameters $(\lambda_1,k_1),\ldots,(\lambda_L,k_L)$. Let t_{ij} represent the jth observation in the ith sample of size n_i, i =

1,...,L. For testing the equality of the scale parameters in presence of common shape parameter k, the competing hypotheses are

$$H_0 : \lambda_1 = \lambda_2 = ... = \lambda_L \ (=\lambda)$$

and

H_1 : at least two λ_i's are different, for all k > 0. For testing the hypothesis of a common shape parameter across all the populations, the competing hypotheses are

$$H_1 : k_1 = k_2 = ... = k_L \ (=k)$$

and

H_2 : not all k_i's are equal, $\lambda_1,...,\lambda_L$ being unspecified.

For the ith, sample, i = 1,...,L, denote the arithmetic mean by

$$\bar{t}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} t_{ij}$$

and

the geometric mean by

$$\tilde{t}_i = \left(\prod_{j=1}^{n_i} t_{ij} \right)^{\frac{1}{n_i}} .$$

Then the log likelihood can be expressed, under the hypothesis H_2 as

$$l = \sum_{i=1}^{L} n_i \left[-k_i \log \lambda_i - \log \Gamma(k_i) + (k_i + 1) \log (\tilde{t}_i) - \frac{\bar{t}_i}{\lambda_i} \right]. \tag{3.2.1}$$

Maximum likelihood equations are obtained by equating the partial derivatives of the log likelihood (3.2.1) with respect to λ_i and k_i, i = 1,...,L, to zero. Accordingly we obtain the maximum likelihood estimates $\tilde{k}_i$ and $\tilde{\lambda}_i$ of k_i and λ_i (i = 1,...,L) by solving

$$n_i \left(\log \tilde{t}_i - \psi(\tilde{k}_i) - \log \tilde{\lambda}_i\right) = 0$$

and

$$\frac{n_i}{\tilde{\lambda}_i}\left(\frac{\bar{t}_i}{\tilde{\lambda}_i} - \tilde{k}_i\right) = 0.$$

The term $\psi(x)$ is the digamma function of x; that is $\psi(x) = \frac{d}{dx} \log \Gamma(x)$. These two

equations yield, for i = 1,...,L,

$$\psi(\tilde{k}_i) - \log \tilde{k}_i = \log (\tilde{t}_i/\bar{t}) \tag{3.2.2}$$

and

$$\tilde{\lambda}_i = \bar{t}_i/\tilde{k}_i \tag{3.2.3}$$

The equation (3.2.2) can be solved iteratively for $\tilde{k}_i$. The maximum likelihood estimates $\tilde{\lambda}_i$ then follows from the equation (3.2.3).

Under the hypothesis H_1, the maximum likelihood estimator, $\tilde{k}$, of the common k is obtained by solving the equation

$$\psi(\tilde{k}) - \log \tilde{k} = \frac{1}{N}\sum_{i=1}^{L} n_i \log (\tilde{t}_i/\bar{t}_i), \tag{3.2.4}$$

iteratively, where $N = \sum n_i$, i = 1,...,L. The maximum likelihood estimator of λ_i, under H_1, then is

$$\tilde{\lambda}_{ic} = \bar{t}_i / \tilde{k}, \quad i = 1,\ldots,L. \tag{3.2.5}$$

Under the hypothesis H_0, the maximum likelihood estimate $\hat{k}$ of k is obtained by solving

$$\psi(\hat{k}) - \log \hat{k} = \frac{1}{N} \sum_{i=1}^{L} n_i \log \bar{t}_i - \log \bar{t}, \tag{3.2.6}$$

where $\bar{t}$ is the overall mean defined as $\bar{t} = (n_1\bar{t}_1 + ... + n_L\bar{t}_L) / N$. The maximum likelihood estimator $\hat{\lambda}$ of λ, under H_0 then , is

$$\hat{\lambda} = \bar{t} / \hat{k}. \tag{3.2.7}$$

3.3. TESTING EQUALITY OF THE SCALE PARAMETERS IN THE PRESENCE OF A COMMON SHAPE PARAMETER

Our interest is to test

H_0: $\lambda_1 = ... = \lambda_L$

against

H_1: not all λ_is are equal with the assumption of common k.

3.3.1. Likelihood Ratio Statistic (LR)

The maximized log likelihood l_1 when all the λ_i's are possibly different is given by

$$l_1 = \sum_{i=1}^{L} n_i \, [(\tilde{k}-1) \log \bar{t}_i - \tilde{k} \log \tilde{\lambda}_{ic} - \log \Gamma(\tilde{k}) - \bar{t}_i / \tilde{\lambda}_{ic} \;].$$

The maximized log likelihood l_0 under the constraint $\lambda_1 = ... = \lambda_L$ is written as

$$l_0 = \sum_{i=1}^{L} n_i[(\hat{k}-1) \log \bar{t}_i - \hat{k} \log \hat{\lambda} - \log \Gamma(\hat{k}) - \bar{t}_i / \hat{\lambda}].$$

As discussed in section 2.15.1, the likelihood ratio statistic LR for testing H_0 against H_1 is given by

$$LR = 2\,(l_1 - l_0) = 2\sum_{i=1}^{L} n_i\,[\,(\hat{k}-\bar{k})(1-\log \bar{t}_i) - \bar{k}\log \tilde{\lambda}_{ic}\,] \qquad (3.3.1)$$
$$+\ 2N\,[\,\hat{k}\log\hat{\lambda} + \log\Gamma(\hat{k}) - \log\Gamma(\bar{k})\,].$$

Under the null hypothesis H_0, the statistic LR is approximately distributed as chisquare with (L-1) degrees of freedom.

3.3.2. Modified Likelihood Ratio Statistics (M and MB)

For a single sample of size n from a gamma distribution having pdf (3.1.1), it is well-known that $2n\bar{X}/\lambda \sim \chi^2(2nk)$ (Shiue and Bain, 1983), where $\bar{X}$ is the sample mean and $\chi^2(v)$ is a chi- squared distribution with v degrees of freedom. Thus, in a multi-sample situation, for the ith sample, $2n_i\bar{t}_i/\lambda_i \sim \chi^2(2n_ik)$, i = 1,...,L. Alternatively, it can be written as $2n_i\,\bar{k}\,\tilde{\lambda}_{ic}/\lambda_i \sim \chi^2(2n_ik)$. Denote $g_i = 2n_i\bar{k}$ and $h_i = 2n_ik$, then $g_i\tilde{\lambda}_{ic}/\lambda_i \sim \chi^2(h_i)$, i=1,...,L. Bartlett (1937) developed a procedure which is a modification of the likelihood ratio test for testing the homogeneity of variances of several groups having normal distribution. Bartlett's test statistic is of the form

$$M = V\log S^2 - \sum_{i=1}^{L} V_i \log S_i^2\ ,$$

where $V = \sum_{i=1}^{L} V_i$, $S^2 = \sum_{i=1}^{L} V_i\, S^2_i/V$ and S^2_i is the sample variance of the ith group with degrees of freedom V_i. When S^2_i is replaced by $g_i\tilde{\lambda}_{ic}/h_i$ the Bartlett's statistic for testing equality of scale parameters is of the form

$$M = h \log \lambda^* - \sum_{i=1}^{L} h_i \log (g_i \tilde{\lambda}_{ic} / h_i),$$

where

$$\lambda^* = \frac{1}{h} \sum_{i=1}^{L} g_i \tilde{\lambda}_{ic} \quad and \quad h = \sum_{i=1}^{L} h_i.$$

The true parameter k is in general unknown. It can be replaced by its maximum likelihood estimate $\tilde{k}$. Then $g_i = h_i = 2n_i\tilde{k}$, i=1,...,L and the statistic M reduces to

$$M = 2 \tilde{k} \sum_{i=1}^{L} n_i \log (\lambda^* / \tilde{\lambda}_{ic}), \quad where\ \lambda^* = \frac{1}{N} \sum_{i=1}^{L} n_i \tilde{\lambda}_{ic}. \tag{3.3.2}$$

Bartlett (1937) showed that the distribution of M is better approximated by $\chi^2(L-1)$ when a small sample correction is used. The corrected statistic (MB) is then given by

$$MB = M/C, \tag{3.3.3}$$

where the correction factor C is given as

$$C = 1 + \frac{1}{6 \tilde{k} (L-1)} \left(\sum_{i=1}^{L} \frac{1}{n_i} - \frac{1}{N} \right).$$

3.3.3 C(α) Statistic (CL)

Let us assume that under the alternative hypothesis $\lambda_i = \lambda + \phi_i$, i = 1,...,L with $\phi_L = 0$. Then testing the null hypothesis of homogeneity of scale parameters reduces to testing H_0: $\phi_i = 0$, for all i, with λ and k treated as nuisance parameters. Tarone (1985), Barnwal and Paul (1988), Paul (1989) and many others applied this technique in similar situations.

Define $\phi = (\phi_1,...,\phi_{L-1})'$ and $\theta = (\theta_1,\theta_2)' = (\lambda,k)'$. Under the above reparametrization, the log likelihood function for the combined sample becomes

$$l = \sum_{i=1}^{L} n_i \left[(k-1)\log \bar{t}_i - k \log (\lambda+\phi_i) - \log \Gamma(k) - \bar{t}_i /(\lambda+\phi_i)\right]. \tag{3.3.4}$$

Based on the likelihood l in (3.3.4), we obtain for $i = 1,...,(L-1)$,

$$\psi_i = \left.\frac{\partial l}{\partial \phi_i}\right|_{\phi=0} = \frac{n_i}{\lambda}\left(\frac{\bar{t}_i}{\lambda} - k\right),$$

$$\eta_1(\theta) = \left.\frac{\partial l}{\partial \theta_1}\right|_{\phi=0} = \left.\frac{\partial l}{\partial \lambda}\right|_{\phi=0} = \frac{N}{\lambda}\left(\frac{\bar{t}}{\lambda}-k\right),$$

and

$$\eta_2(\theta) = \left.\frac{\partial l}{\partial \theta_2}\right|_{\phi=0} = \left.\frac{\partial l}{\partial k}\right|_{\phi=0} = \sum_{i=1}^{L} n_i \log \bar{t}_i - N\,[\psi(k)-\log \lambda].$$

Under the null hypothesis, the expected values of the negative mixed partial derivatives are

$$A_{ij} = E\left(-\left.\frac{\partial^2 l}{\partial\phi_i\partial\phi_j}\right|_{\phi=0}\right) = \begin{cases} \dfrac{n_i k}{\lambda^2}, & 1 \le i = j \le (L-1) \\ 0, & 1 \le i \ne j \le (L-1) \end{cases},$$

$$B_{ij} = E\left(- \frac{\partial^2 l}{\partial\phi_i \partial\theta_j}\bigg|_{\phi=0} \right) = \begin{cases} \dfrac{n_i\, k}{\lambda^2} \;, & j = 1;\; i = 1,\ldots,(L-1) \\ \dfrac{n_i}{\lambda} \;, & j = 2;\; i = 1,\ldots,(L-1) \end{cases},$$

and

$$C_{jj'} = E\left(- \frac{\partial^2 l}{\partial\theta_j\, \partial\theta_{j'}}\bigg|_{\phi=0} \right) = \begin{cases} N\,\dfrac{k}{\lambda^2} \;, & j = j' = 1 \\ N\,\psi'(k) \;, & j = j' = 2 \\ \dfrac{n}{\lambda} \;, & j \neq j';\; j, j' = 1, 2. \end{cases}$$

Now, the parameters λ and k in ψ_i, η_1, η_2, A_{ij}, B_{ij} and $C_{jj'}$ are replaced by their MLEs. Then, following the general theory in section 2.15.2, the $C(\alpha)$ statistic is $CL = \psi' V^{-1} \psi$, where $V = A - B\, C^{-1} B'$. After considerable steps of algebra the (i,j)th element of V is

$$V_{ij} = \begin{cases} \dfrac{N\,k\,p_i\,(1-p_i)}{\lambda^2} \;, & i=j, \\ \dfrac{-N\,k\,p_i\,p_j}{\lambda^2} \;, & i\neq j, \end{cases}$$

i,j = 1,...,(L-1), where $p_i = n_i/N$, i = 1,...,(L-1). After simplification, we obtain

$$CL = \hat{k} \sum_{i=1}^{L} n_i \left(\bar{t}_i / \bar{t} - 1 \right)^2. \tag{3.3.5}$$

When L = 2, the square root of the statistic CL is identical to the $C(\alpha)$ statistic given by equation (27) of Moran (1970).

3.3.4 Extremal Scale Parameter Ratio (EP)

In two sample case, Shiue and Bain (1983) studied a one sided test based on the ratio of two sample means. As we described earlier for a known k, $2n_i\bar{t}_i/k \sim \chi^2(2n_ik)$, i = 1,2. Then,

$$F = \frac{\dfrac{2\,n_1\bar{t}_1}{k}/2n_1k}{\dfrac{2\,n_2\bar{t}_2}{k}/2n_2k} = \frac{\bar{t}_1}{\bar{t}_2} \sim F\,(\,2n_1k, 2n_2k\,) \tag{3.3.6}$$

In practice, k is an unknown parameter. Shiue and Bain (1983) showed that the distribution of $\bar{t}_1/\bar{t}_2$, under H_0, is approximately $F(2n_1\hat{k},\ 2n_2\hat{k})$ when k is replaced by its maximum likelihood estimator $\hat{k}$. They also showed that the actual size of the approximate test given by $P\,(k,\alpha) = P[\ \bar{t}_1/\bar{t}_2 < F_{\alpha},(2n_1\hat{k},\ 2n_2\hat{k})\ ]$ is free of the common scale parameter λ and depends little on k.

Now, for two samples, we consider a two-sided test as this is a special case of the general hypothesis for testing homogeneity of L scale parameters. We reject the null hypothesis if $F = \bar{t}_1/\bar{t}_2 < F_{(1-\alpha/2)},(2n_1\hat{k},\ 2n_2\hat{k})$ or $F = \bar{t}_1/\bar{t}_2 > F_{\alpha/2},(2n_1\hat{k},\ 2n_2\hat{k})$.
An alternative test statistic can be constructed as

$$EP = \frac{\max\limits_i\{\bar{t}_i\}}{\min\limits_i\{\bar{t}_i\}} = \frac{\max\limits_i\{\bar{t}_i/\hat{k}\}}{\min\limits_i\{\bar{t}_i/\hat{k}\}} = \frac{\max\limits_i\{\hat{\lambda}_{ic}\ \}}{\min\limits_i\{\hat{\lambda}_{ic}\ \}}. \tag{3.3.7}$$

The statistic EP, known as extremal scale parameter ratio statistic, is in general applicable for testing the equality of several scale parameters. The null hypothesis would be rejected in favour of the alternative hypothesis for large values of EP. McCool(1979) proposed an analogous statistic for testing equality of scale parameters of several extreme value populations. For two samples of equal sizes ($n_1 = n_2 = n$) with known k, the statistic EP $= \mathrm{Max}(\bar{t}_1, \bar{t}_2)/\mathrm{Min}(\bar{t}_1, \bar{t}_2)$ has a truncated F(2nk, 2nk) distribution, as shown below.

We know that for known k, $X_i = 2n_i t_i / k \sim \chi^2(2n_i k)$, i = 1,2. If $n_1 = n_2 = n$, then X_1 and X_2 are independently and identically distributed as $\chi^2(h)$, where h = 2nk. Suppose $X_{(1)}$ and $X_{(2)}$ are the ordered statistics from X_1 and X_2 and let $R = X_{(2)}/X_{(1)}$, then the distribution of R is

$$2 \int_0^\infty x \left[\frac{e^{-Rx/2} \, (Rx)^{h/2-1} \; e^{-x/2} \, x^{h/2-1}}{2^{h/2} \, \Gamma(h/2) \; 2^{h/2} \, \Gamma(h/2)} \right] dx$$

$$= \frac{2}{\mathrm{B}(h/2, \, h/2)} \, \frac{R^{h/2-1}}{(1+R)^h}, \quad r \geq 1.$$

This is the density of a truncated F distribution with degrees of freedom (h,h). The 100α percentage point C of the distribution of R can be obtained from

$$F_c(h,h) = 1-\alpha/2, \tag{3.3.8}$$

where $F_c(h,h)$ is the cumulative probability of the F distribution with degrees of freedom (h,h). The value of C from (3.3.8) can be obtained using appropriate subroutine from IMSL or NAG. For $L > 2$, $n_1 = \ldots = n_L$ and k known, the distribution of EP has a complicated form and no closed form solution for the critical value exists. In other situations, the distribution of EP is not known. Therefore, for $L > 2$, the percentage points need to be obtained by Monte Carlo simulations. Now, we discuss the practical situation that k is not known. In this case, if k is replaced by the maximum likelihood estimate $\tilde{k}$, then, for given α, the actual size of the test is $P(k,\alpha) = P[\ EP > EP(\tilde{k},\alpha)\ ]$, where $EP(\tilde{k},\alpha)$ is the $100\alpha\%$ point of the distribution of the statistic EP. Obviously $P(k,\alpha)$ would be independent of k and λ. However, it is unclear if the critical value $EP(\tilde{k},\alpha)$ remains unchanged as k and λ vary. To investigate the behaviour of $EP(\tilde{k},\alpha)$ for various values of k and λ we conducted a small scale simulation study. This study was limited to L = 3; $\alpha = 0.01, 0.05, 0.10$; $(n_1, n_2, n_3) = (10, 10, 10)$ and $\lambda = 0.1, 1.5$. For the evaluation of $EP(\tilde{k},\alpha)$, 10,000 random samples from two parameter gamma distribution were generated by using the IMSL subroutine RNGAM. The simulations showed that EP is independent of λ. So we reported results for only $\lambda = 0.1$ in Table 3.1.

Table 3.1

Values of $EP(\tilde{k},\alpha)$ for $L = 3$; $\lambda = 0.1$; $(n_1,n_2,n_3) = (10,10,10)$.

k	0.01	0.05	0.10
0.5	7.5471	4.8576	3.8943
1.0	3.8974	2.9598	2.5724
1.5	3.0985	2.4195	2.1568
2.0	2.5784	2.1343	1.9415
2.5	2.3683	1.9872	1.8049
3.0	2.1218	1.8309	1.7003
4.0	1.9207	1.6987	1.5902
6.0	1.7083	1.5349	1.4598
10.0	1.5140	1.4024	1.3446

Table 3.1 shows that the critical values $EP(\tilde{k},\alpha)$ decrease as k increases. This behaviour of $EP(\tilde{k},\alpha)$ has been observed in various combinations of sample sizes.

3.4 SIMULATION STUDY

Performance of the statistics LR, M, MB, CL and EP developed in section 3.3 are compared in terms of size and power by using Monte Carlo simulations. The empirical levels are computed based on 2000 replications. Empirical levels were found to be independent of common scale parameter λ, so we present results for $\lambda = 0.1$. The empirical levels were calculated for various combinations of sample sizes; number of groups $L = 2, 3$; common shape parameter $k = 1.0, 1.5$ and the nominal levels $\alpha = 0.01, 0.05, 0.10$. The results are presented in Tables 3.2 and 3.3. For $L = 2$, empirical level of the statistic EP was computed based on the distribution of $F = \bar{t}_1 / \bar{t}_2$ discussed in section 3.3.4. For $L > 2$, EP is not included in the study of empirical size as it's exact distribution is not known.

For power comparison, first, the critical values of the statistics LR, M, MB, CL and EP were computed from the empirical distributions, based on 10,000 replications. Using these critical values, power of the above five statistics for various combinations of sample sizes was calculated based on 2000 replications for $L = 2, 3$; $k = 1.0, 1.5$ and $\alpha = 0.01, 0.05, 0.10$. The results are reported in Tables 3.4 through 3.9.

Results

From Tables 3.2 and 3.3, we see that the statistic CL holds nominal level well for $\alpha = 0.05$ and 0.10, although it is slightly conservative for $\alpha = 0.01$ and small sample sizes. All other statistics are in general liberal. Note that two times the standard error of the probabilities based on $\alpha = 0.01, 0.05$ and 0.10 are respectively 0.005, 0.010 and 0.013. Empirical levels less than α - 2 (standard error) are termed as conservative and those greater than α + 2 (standard error) are termed as liberal. Tables 3.4 through 3.9 show that for equal sample size situations power of all the statistics is similar except for the statistic EP which is slightly more powerful. Further, power of all the statistics increases as k increases. Performance of the statistics LR, M and MB is similar for the case of unequal sample sizes. The behaviour of the statistic CL and that of the statistic

EP is just opposite in unequal sample size situations. When $\lambda_1 < \lambda_2 < \lambda_3$ the statistic CL has least power and the statistic EP is most powerful in the case of $n_1 < n_2 < n_3$ and the statistic CL is most powerful and EP is least powerful in the case of $n_1 > n_2 > n_3$. In both cases, some of the n_i's may be equal. The power performance is similar for the nominal level $\alpha = 0.01, 0.05$ and 0.10.

Since the statistics LR, M and MB are, in general, liberal and they do not show power advantage over the other two statistics, we report on a small scale simulation study only for the statistics CL and EP for $L = 5, 10$; $\alpha = 0.01, 0.05, 0.10$ and $k = 1.5$; and for a number of combinations of sample sizes. Empirical levels of the statistic CL based on its asymptotic chi square distribution are given in Table 3.10. Based on the empirically computed percentage points, power values of the statistics CL and EP are obtained and are given in Table 3.11. Conclusions for $L = 5, 10$ in Table 3.10 and 3.11 for the statistics CL and EP are the same as those for $L = 2, 3$ in Tables 3.3 through 3.9.

3.5. TESTING THE HOMOGENEITY OF SHAPE PARAMETERS OF SEVERAL GAMMA DISTRIBUTIONS

The procedures for testing the equality of scale parameters in section 3.3 are based on the assumption that the shape parameters of the populations are equal. However, this assumption should be checked before testing the homogeneity of scale parameters. For this purpose we develop two test procedures in this section. The competing hypotheses are

$$H_1: \; k_1 = \ldots = k_L \; (= k)$$

and

H_2: not all k's are equal, $\lambda_1, \ldots, \lambda_L$ being unspecified scale parameters.

The test statistics are derived as follows:

3.5.1. Likelihood Ratio Statistic (LRk)

Using the maximum likelihood estimators of the parameters under the null and alternative hypotheses given in section 3.2, the maximum log likelihood function, under H_2, is

$$l_2 = \sum_{i=1}^{L} n_i \left[(\bar{k}_i - 1) \log \bar{t}_i - \frac{\bar{t}_i}{\bar{\lambda}_i} - \bar{k}_i \log \bar{\lambda}_i - \log\Gamma(\bar{k}_i) \right]$$

and under H_1, the maximum log likelihood function is

$$l_1(\bar{\lambda},\bar{k}) = \sum_{i=1}^{L} n_i \left[(\bar{k}-1) \log \bar{t}_i - \frac{\bar{t}_i}{\bar{\lambda}_{ic}} - \bar{k} \log \bar{\lambda}_{ic} - \log \Gamma(\bar{k}) \right]$$

Thus, the log likelihood ratio statistic (LRk), as discussed in section 2.15.1, is given by

$$\begin{aligned} LRk &= 2 \left[l_2(\bar{\lambda},\bar{k}) - l_1(\bar{\lambda},\bar{k}) \right] \\ &= 2 \sum_{i=1}^{L} n_i \left\{ \bar{k} \log \bar{\lambda}_{ic} - \bar{k}_i \log \bar{\lambda}_i + \log \left(\Gamma(\bar{k}) / \Gamma(\bar{k}_i)\right) \right\} \qquad (3.5.1) \\ &+ 2 \sum_{i=1}^{L} n_i \left\{ (\bar{k} - \bar{k}_i) (1 - \log \bar{t}_i) \right\} . \end{aligned}$$

Under the null hypothesis H_1, distribution of the statistic LRk is approximately distributed as chi square with (L-1) degrees of freedom.

3.5.2. C(α) Statistic (CLk)

We follow the procedure as stated in section 3.3.3. Reparametrize the shape parameters as $k_i = k + \phi_i$, $i = 1,...,L$ with $\phi_L = 0$. Define $\phi = (\phi_1,...,\phi_{L-1})'$ and $\theta = (\theta_1,...,\theta_L,\theta_{L+1})' = (\lambda_1,...,\lambda_L,k)'$.

Using the notations described earlier

$$\psi_i = \left.\frac{\partial l}{\partial \phi_i}\right|_{\phi=0}, \qquad i = 1,...,(L-1)$$

and

$$\delta_j = \left.\frac{\partial l}{\partial \theta_j}\right|_{\phi=0}, \qquad j = 1,...,(L+1).$$

Define

$$T_i(\theta) = \psi_i(\theta) - \sum_{j=1}^{L+1} \beta_{ij}\, \delta_j(\theta),$$

where β_{ij}, $i = 1,...,(L-1)$ is the partial regression coefficient of ψ_i on δ_j, $j = 1,...,(L+1)$. As stated earlier, when θ in $T_i(\theta)$ is replaced by its MLE, the coefficient β_{ij} vanishes for i = 1,...,(L-1), and thus for i = 1,....,(L-1), $T_i(\bar{\theta}) = \psi_i(\bar{\theta})$, where the MLE $\bar{\theta}$ of θ is given by $\bar{\theta} = (\tilde{\lambda}_{1c},...,\tilde{\lambda}_{Lc},\bar{k})'$ obtained in section 3.2. Now, the variance covariance of T = ($T_1(\theta),...,T_{L-1}(\theta)$)' is given by $C - E\,F^{-1}\,E'$, where the (i,j) th elements of C, E and F are as follows:

$$C_{ij} = E\left(- \left.\frac{\partial^2 l}{\partial\phi_i \partial\phi_j}\right|_{\phi=0}\right) = \begin{cases} n_i\, \psi'(k) & , \quad i = j = 1,...,(L-1) \\ 0 & , \quad i \neq j = 1,...,(L-1) \end{cases},$$

$$e_{ij} = E\left(\left.\frac{\partial^2 l}{\partial\phi_i \partial\theta_l}\right|_{\phi=0}\right) = \begin{cases} \dfrac{n_i}{\lambda_i}, & i = l = 1,\dots,(L-1) \\ 0, & i \neq l;\ l = 1,\dots,L \\ & i = 1,\dots,(L-1) \\ n_i\,\psi'(k), & l = L+1;\ i = 1,\dots,(L-1) \end{cases}$$

and

$$F_{l\,l'} = -E\left(\left.\frac{\partial^2 l}{\partial\theta_l \partial\theta_{l'}}\right|_{\phi=0}\right) = \begin{cases} \dfrac{n_l k}{\lambda^2{}_l}, & l = l' = 1,\dots,L \\ 0, & l \neq l' = 1,\dots,L \\ \dfrac{n_l}{\lambda_l}, & l = L+1;\ l' = 1,\dots,L \\ & l' = L+1;\ l = 1,\dots,L \\ N\,\psi'(k), & l = l' = (L+1) \end{cases}$$

The $C(\alpha)$ statistic has the following form $CLk = T'(\,C - EF^{-1}E'\,)T$. After some considerable steps of algebra, and replacing θ by its MLE $\tilde{\theta}$, the $C(\alpha)$ statistic is obtained as

$$CLk = \frac{\bar{k}}{(\bar{k}\,\psi'(\bar{k}) - 1\,)} \sum_{i=1}^{L} \frac{T_i^2}{n_i},$$

where

$$T_i = \psi_i(\tilde{\theta}) = n_i\,[\,\log \bar{t}_i - \log \tilde{\lambda}_{ic} - \psi(\bar{k})\,];$$

that is

$$CLk = \frac{\bar{k}}{(\bar{k}\psi'(\bar{k}) - 1)} \sum_{i=1}^{L} n_i [\log \bar{r}_i - \log \hat{\lambda}_{ic} - \psi(\bar{k})]^2 , \tag{3.5.2}$$

which is asymptotically distributed as chisquare with (L-1) degrees of freedom.

3.6. SIMULATION STUDY

A small scale simulation study was conducted to compare the size and power of the statistics LRk and CLk. For both size and power comparison, we considered $L = 2, 3$; $\alpha = 0.01, 0.05, 0.10$. Empirical levels and power are independent of the scale parameter values, so we chose $\lambda = (\lambda_1,\lambda_2,\lambda_3) = (0.1,0.2,0.3)$. The common value of k was chosen as 0.5. Each experiment for empirical levels was based on 2000 replications. Empirical levels are given in Table 3.12. For power comparison, first empirical percentage points were obtained based on 10000 replications and then using these percentage points powers of these two statistics were calculated based on 2000 replications and are reported in Tables 3.13, 3.14 and 3.15.

Results

From Table 3.12, we can see that the statistic LRk is slightly liberal. The statistic CLk holds the nominal level well, except in small sample situations for $L = 2$. When L increases CLk maintains the significance level quite well. Note that two times the standard error of the probabilities reported is approximately 0.005, 0.010 and 0.013 for $\alpha = 0.01$, 0.05 and 0.10. Table 3.13 shows that for two groups with equal sample sizes, both statistics have similar power. For $L = 3$, power of the statistic CLk is consistently better than that of LRk, except for the situation, where $n_1 \geq n_2 > n_3$ or $n_1 > n_2 \geq n_3$ with increasing k_i's$(i = 1,\ldots,L)$.

3.7. EXAMPLES

Example 1: The data studied by Simpson (1972) refer to the radar evaluated rainfall data from 52 South Florida cumulus clouds, 26 seeded and 26 control clouds lifetimes.

seeded:	129.6, 31.4, 2745.6, 489.1, 430.0, 302.8, 119.0, 4.1, 92.4, 17.5, 200.7, 274.7, 274.7, 7.7, 1656.0, 978.0, 198.6, 703.4, 1697.8, 334.1, 118.3, 255.0, 115.3, 242.5, 32.7, 40.6.
control:	26.1, 26.3, 87.0, 95.0, 372.4, 0.1, 17.3, 24.4, 11.5, 321.2, 68.5, 81.2, 47.3, 28.6, 830.1, 345.5, 1202.6, 36.6, 4.9, 4.9, 41.1, 29.0, 163.0, 244.3, 147.8, 21.7.

We first test the equality of the shape parameters. In this case, we obtain $\hat{\lambda}_1 = 767.342$, $\hat{\lambda}_2 = 285.687$, $\hat{k} = 0.576$ and CLk =0.360. This shows no evidence against the assumption that the shape parameters are equal. Since $n_1 = n_2$ we use the statistic CL for testing equality of the scale parameters. The value of CL is 5.729 with p- value 0.017. Note that for this example, the F statistic and the EP statistic are both (767.342/285.687 =) 2.686. The common degree of freedom of F and EP is 2(26) (0.576) = 29.952 with p- value 0.009. Thus, the conclusion based on F or EP is the same as that based on CL. That is, there is evidence of, possibly, different means for the two groups.

Example 2: For illustrative purposes we simulated three samples from gamma distributions with parameters (0.1,0.5), (0.2,0.5), (0.3,0.5). Sample sizes are taken as

$(n_1,n_2,n_3) = (10,20,30)$. The data are:

sample 1: 0.00477, 0.00521, 0.00050, 0.01137, 0.00352, 0.00823, 0.05301, 0.05477, 0.03841, 0.16865.

sample 2: 0.00001, 0.08749, 0.10031, 0.02197, 0.03175, 0.07410, 0.19230, 0.37005, 0.12567, 0.34200, 0.03030, 0.01902, 0.01487, 0.02643, 0.00510,0.23954, 0.00972, 0.05956, 0.15482, 0.02788.

sample 3: 0.25047, 0.00017, 0.02383, 0.07534, 0.63579, 0.33650, 0.02926, 0.00229, 0.80316, 0.04149, 0.02522, 0.00013, 0.54039, 0.09198, 0.00544, 0.05421, 0.23740, 0.20021, 0.07539, 0.20822, 0.19611, 0.32148, 0.01334, 0.06759, 0.84330, 0.43629, 0.00821, 0.36822, 0.00972, 0.08071.

Now, for this data set, $\tilde{\lambda}_1 = 0.063$, $\tilde{\lambda}_2 = 0.175$, $\tilde{\lambda}_3 = 0.360$, $\tilde{k} = 0.554$. The value of CLk, on 2 degrees of freedom, is 0.326 indicating strong evidence in favour of the assumption of common k. Also, in this example $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \tilde{\lambda}_3$ and $n_1 < n_2 < n_3$, so, we use the statistic EP to test for the equality of common scale parameters. The estimate of the common λ is $\tilde{\lambda} = 0.279$ and that of the common k is $\tilde{k} = 0.493$. The value of EP is 5.723 with p - value = 0.007. The p- value was calculated from the empirical distribution of the statistic EP calculated from 10,000 sets of three samples ($n_1 = 10$, $n_2 = 20$, $n_3 = 30$) which were simulated with common $\lambda = 0.279$ and common $k = 0.493$. The p- value indicates evidence that the scale parameters are not equal. Note that the value of CL is 6.592 with p- value 0.010. In this example also conclusion based on CL is the same as that based on EP.

Table 3.2: Empirical levels (%) of the test statistics LR, M, MB, CL and EP based on 2000 replications. L = 2.

		k = 1.0			k = 1.5		
		α			α		
Tests	n_1,n_2	10.0	5.0	1.0	10.0	5.0	1.0
LR	10,10	11.9	6.3	1.8	12.8	5.8	1.4
M		12.7	7.3	2.5	13.4	7.0	2.0
MB		12.4	7.2	2.4	13.4	6.7	1.9
CL		9.3	4.3	0.4	10.4	4.2	0.4
EP		12.0	6.6	1.5	15.4	9.1	2.7
LR	20,20	11.0	5.8	1.2	12.3	5.9	1.5
M		11.4	6.6	1.6	12.8	6.6	1.8
MB		11.3	6.2	1.5	12.8	6.6	1.8
CL		10.3	4.8	0.6	11.3	5.1	0.8
EP		11.2	5.8	1.1	10.1	4.5	0.9
LR	10,20	12.1	6.4	1.8	12.2	6.5	1.6
M		12.8	7.2	2.2	12.6	7.5	2.0
MB		12.5	7.0	2.2	12.5	7.3	2.0
CL		10.0	4.9	1.2	10.4	5.5	0.7
EP		12.8	7.3	1.9	16.8	10.8	3.4
LR	15,20	10.8	5.8	1.1	12.2	6.8	1.7
M		11.4	6.3	1.5	13.0	7.4	2.0
MB		11.1	6.2	1.4	12.8	7.3	2.0
CL		9.7	4.6	0.7	11.4	5.7	0.9
EP		11.5	6.1	1.6	12.5	6.7	1.7
LR	20,10	11.8	7.1	1.8	11.3	6.2	1.1
M		12.4	7.7	2.3	11.7	7.2	1.1
MB		12.1	7.5	2.2	11.6	7.1	1.4
CL		10.3	5.2	0.9	10.0	4.8	0.5
EP		10.5	5.9	1.3	8.4	3.9	0.6

Table 3.3: Empirical levels (%) of the test statistics LR, M, MB, and CL based on 2000 replications, L = 3.

		k = 1.0			k = 1.5		
			α			α	
Tests	n_1,n_2,n_3	10.0	5.0	1.0	10.0	5.0	1.0
LR	10,10,10	12.4	6.7	1.6	12.4	7.0	1.5
M		14.1	8.2	2.6	13.8	8.1	2.5
MB		13.6	7.9	2.5	13.4	8.0	2.4
CL		9.3	3.8	0.7	9.6	4.8	0.7
LR	20,20,20	11.5	5.9	1.3	11.8	6.9	1.3
M		12.4	6.6	1.7	12.7	7.6	1.7
MB		12.2	6.5	1.5	12.5	7.5	1.7
CL		9.9	4.8	0.8	10.4	5.3	0.8
LR	10,10,20	11.8	6.1	1.7	11.3	6.0	1.2
M		13.5	7.2	2.2	12.9	7.0	1.8
MB		13.0	7.0	2.1	12.7	7.0	1.7
CL		9.7	4.3	0.8	9.5	4.2	0.7
LR	15,15,20	11.2	5.4	1.3	11.8	6.5	1.1
M		11.9	6.4	1.6	13.1	7.8	1.5
MB		11.8	6.3	1.6	12.8	7.7	1.6
CL		9.2	4.3	0.9	10.3	4.7	0.8
LR	20,20,10	12.2	6.3	1.6	11.1	6.3	1.5
M		13.9	7.2	2.2	12.2	6.9	2.1
MB		13.4	7.0	2.0	11.9	6.8	2.1
CL		9.9	4.6	1.1	9.4	5.3	1.0
EP		10.5	5.9	1.3	8.4	3.9	0.6

Table 3.4: Empirical power (%) of the statistics LR, M, MB, CL and EP; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.01$, $k = 1.0, 1.5$; $L = 2$.

		k = 1.0				k = 1.5			
		(λ_1,λ_2)				(λ_1,λ_2)			
Tests	n_1,n_2	(.1,.15)	(.1,.2)	(.1,.25)	(.1,.3)	(.1,.15)	(.1,.2)	(.1,.25)	(.1,.3)
LR	10,10	3.0	10.1	21.4	33.2	5.9	18.8	35.5	53.2
M		3.1	9.9	20.9	32.9	5.8	18.5	35.4	52.6
MB		3.1	9.9	20.9	32.7	5.9	18.4	35.5	52.4
CL		3.0	9.5	18.9	30.4	5.6	17.8	34.1	50.6
EP		3.7	12.3	25.9	40.8	7.3	23.9	46.1	64.3
LR	20,20	8.7	29.5	56.0	76.7	13.7	46.4	76.8	91.9
M		8.7	29.5	55.9	76.6	13.6	46.3	76.7	92.0
MB		8.7	29.6	55.9	76.6	13.7	46.2	76.7	91.9
CL		8.7	29.4	55.5	76.4	13.8	46.0	75.1	91.7
EP		9.3	33.7	62.2	81.3	14.1	49.7	80.8	94.1
LR	10,20	5.3	16.3	32.7	49.0	7.5	26.5	49.9	70.1
M		5.2	16.3	32.6	48.9	7.3	26.2	49.5	69.8
MB		5.3	16.5	32.5	48.7	7.5	26.1	49.4	69.7
CL		1.1	3.9	8.4	14.6	3.6	13.9	29.2	47.3
EP		6.7	21.3	38.7	56.5	9.3	32.6	58.4	79.2
LR	15,20	6.9	25.5	46.7	65.3	12.0	39.0	68.3	86.5
M		6.7	25.1	46.4	65.1	11.9	38.9	68.3	86.5
MB		6.8	25.0	46.4	64.9	12.0	38.8	68.2	86.5
CL		5.5	20.9	40.5	59.3	9.7	33.7	62.7	82.4
EP		7.4	27.8	51.0	71.8	13.4	43.5	75.4	90.9
LR	20,10	5.6	19.2	37.4	54.8	10.5	35.1	61.2	79.0
M		5.5	19.1	37.2	54.6	10.4	34.7	60.9	78.6
MB		5.5	19.1	37.1	54.5	10.5	34.6	61.0	78.6
CL		7.8	26.5	46.6	63.5	15.2	44.0	69.7	85.1
EP		4.1	17.0	35.4	53.6	8.2	32.9	60.2	78.6

Table 3.5: Empirical power (%) of the statistics LR, M, MB, CL and EP; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.05$, $k = 1.0, 1.5$; $L = 2$.

		k = 1.0				k = 1.5			
		(λ_1,λ_2)				(λ_1,λ_2)			
Tests	n_1,n_2	(.1,.15)	(.1,.2)	(.1,.25)	(.1,.3)	(.1,.15)	(.1,.2)	(.1,.25)	(.1,.3)
LR	10,10	12.7	30.7	47.6	62.4	18.7	42.2	65.2	79.6
M		12.6	30.6	47.3	62.3	18.7	42.0	65.0	79.3
MB		12.7	30.5	47.3	62.1	18.7	41.9	64.9	79.1
CL		12.6	32.8	46.7	61.4	18.6	41.7	64.6	78.9
EP		14.3	32.1	51.8	62.3	19.0	45.7	68.3	84.2
LR	20,20	23.2	56.1	80.3	92.4	32.7	74.2	93.1	98.8
M		23.1	56.1	80.2	92.4	32.7	74.2	93.1	98.7
MB		23.2	56.2	80.4	92.4	32.7	74.2	93.1	98.8
CL		23.2	56.0	80.3	92.4	32.7	74.4	93.2	98.8
EP		24.7	58.6	83.1	93.3	34.0	76.3	94.0	98.7
LR	10,20	16.1	38.1	59.4	74.3	22.3	52.1	76.6	90.6
M		16.1	38.0	59.4	74.3	23.0	52.6	77.3	90.9
MB		16.1	38.0	59.5	74.4	22.3	52.0	76.5	90.5
CL		11.5	28.9	48.4	65.2	17.0	44.0	68.9	85.1
EP		19.4	42.2	64.6	80.0	25.5	57.5	81.7	93.5
LR	15,20	21.5	49.2	72.4	86.6	27.8	66.1	87.9	96.5
M		21.4	49.1	72.4	86.5	27.7	65.9	87.9	96.4
MB		21.6	49.1	72.4	86.6	27.8	65.9	87.9	96.5
CL		19.7	45.7	69.3	84.2	25.8	63.7	86.2	95.7
EP		22.7	52.0	75.6	88.9	29.6	70.2	90.8	97.3
LR	20,10	17.4	40.8	61.9	78.1	25.5	58.2	80.1	91.7
M		17.3	40.7	61.9	78.1	25.4	58.1	80.0	91.7
MB		17.5	40.9	62.1	78.2	25.2	58.3	79.9	91.7
CL		21.8	47.4	68.4	82.6	29.5	63.3	83.9	93.7
EP		16.1	39.8	61.5	77.6	24.4	56.6	80.2	92.4

Table 3.6: Empirical power (%) of the statistics LR, M, MB, CL and EP; Critical values based on 10,000 replications; Power based on 2000 replications. $\alpha = 0.10$, $k = 1.0, 1.5$; $L = 2$.

		k = 1.0				k = 1.5			
		(λ_1,λ_2)				(λ_1,λ_2)			
Tests	n_1,n_2	(.1,.15)	(.1,.2)	(.1,.25)	(.1,.3)	(.1,.15)	(.1,.2)	(.1,.25)	(.1,.3)
LR	10,10	24.0	44.6	62.8	76.7	27.7	55.5	75.6	88.1
M		24.0	44.5	62.7	76.6	27.7	55.4	75.6	88.0
MB		24.0	44.4	62.7	76.5	27.8	55.5	75.4	87.8
CL		24.0	44.2	62.5	76.3	27.6	55.5	75.3	87.7
EP		24.2	46.9	64.9	78.5	28.8	57.1	79.4	87.1
LR	20,20	34.5	69.4	88.8	96.3	44.5	82.8	96.5	99.3
M		34.4	69.4	88.8	96.2	44.4	82.7	96.4	99.3
MB		34.4	69.4	88.8	96.3	44.4	82.8	96.4	99.3
CL		34.4	69.4	88.8	96.2	44.4	82.8	96.4	99.3
EP		35.5	70.7	89.9	96.4	45.4	83.9	96.9	99.5
LR	10,20	26.5	51.2	72.4	85.6	33.2	66.7	86.6	96.1
M		26.4	51.1	72.3	85.5	33.1	66.7	86.5	95.9
MB		26.3	51.0	72.4	85.5	33.2	66.7	86.5	96.0
CL		22.2	46.1	67.3	81.1	29.9	62.1	83.5	93.8
EP		29.1	54.1	74.9	88.2	35.9	70.6	89.5	96.6
LR	15,20	31.8	62.2	82.6	93.4	39.9	78.3	93.7	98.5
M		31.7	62.1	82.5	93.3	39.8	78.2	93.6	98.5
MB		31.8	62.1	82.6	93.4	39.9	78.3	93.7	98.5
CL		30.1	59.8	81.2	92.5	37.7	76.4	93.0	98.2
EP		33.4	64.5	85.1	94.5	40.9	80.1	94.5	98.9
LR	20,10	27.9	54.3	74.7	86.6	35.1	68.6	87.6	95.4
M		27.9	54.3	74.7	86.6	35.1	68.4	87.5	95.4
MB		27.9	54.3	74.6	86.6	35.0	68.4	87.4	95.4
CL		31.3	58.6	78.1	88.9	39.1	72.5	89.6	96.3
EP		26.7	53.6	73.3	86.2	34.4	68.9	86.9	95.4

Table 3.7: Empirical power (%) of the test statistics LR, M, MB, CL and EP; critical values based on 10,000 replications; power based on 2000 replications. L = 3, $\alpha = 0.01$; k = 1.5.

		$(\lambda_1,\lambda_2,\lambda_3)$				
Tests	n_1,n_2,n_3	(.1,.12,.12)	(.1,.1,.15)	(.1,.2,.2)	(.1,.2,.25)	(.1,.2,.3)
LR	10,10,10	1.75	5.40	17.30	29.70	44.60
M		1.75	5.30	17.05	29.15	43.95
MB		1.75	5.25	16.95	29.00	43.75
CL		1.20	5.90	9.95	18.05	33.60
EP		1.55	5.45	18.80	32.65	50.65
LR	20,20,20	2.30	13.20	48.35	70.75	87.60
M		2.30	13.30	48.45	70.55	87.65
MB		2.35	13.30	48.55	70.45	87.65
CL		2.20	15.95	36.05	60.05	83.05
EP		2.80	14.00	54.05	76.95	91.20
LR	10,10,20	1.80	8.40	20.30	37.90	58.40
M		1.80	8.40	20.20	37.75	58.30
MB		1.80	8.40	20.20	37.80	58.25
CL		1.10	4.65	7.40	14.30	30.20
EP		2.05	6.95	25.00	42.05	62.65
LR	15,15,20	2.05	12.25	34.60	56.55	77.30
M		2.00	12.15	34.45	56.25	77.10
MB		2.00	12.10	34.35	56.10	77.00
CL		1.75	11.90	22.35	42.10	66.85
EP		2.20	10.95	41.25	62.90	83.15
LR	20,20,10	2.45	7.45	42.80	58.70	73.05
M		2.45	7.40	42.65	58.55	72.95
MB		2.45	7.35	42.50	58.40	72.90
CL		2.35	11.40	34.10	52.05	69.50
EP		1.55	7.20	36.30	56.95	73.55

Table 3.8: Empirical power (%) of the test statistics LR, M, MB, CL and EP; critical values based on 10,000 replications; power based on 2000 replications. $L = 3$, $\alpha = 0.05$; $k = 1.5$.

Tests	n_1,n_2,n_3	($\lambda_1,\lambda_2,\lambda_3$) (.1,.12,.12)	(.1,.1,.15)	(.1,.2,.2)	(.1,.2,.25)	(.1,.2,.3)
LR	10,10,10	7.40	17.45	42.05	57.60	71.35
M		7.45	17.45	41.90	57.50	71.25
MB		7.45	17.50	41.80	57.15	71.00
CL		7.40	19.30	36.00	50.15	65.00
EP		7.20	17.65	46.45	62.60	76.35
LR	20,20,20	9.55	34.25	76.75	90.70	97.35
M		9.55	34.15	76.80	90.75	97.35
MB		9.60	34.15	76.80	90.65	97.35
CL		9.65	36.10	71.30	88.55	96.60
EP		9.65	32.60	78.55	92.05	97.80
LR	10,10,20	8.00	23.40	45.95	66.45	83.10
M		8.00	23.45	46.05	66.45	83.05
MB		9.95	23.40	45.95	66.10	82.95
CL		6.95	19.35	33.95	53.20	74.25
EP		8.55	21.15	53.45	71.40	85.15
LR	15,15,20	7.50	29.45	60.90	80.75	93.00
M		7.50	29.30	60.85	80.65	93.00
MB		7.45	29.25	60.80	80.65	92.90
CL		6.75	29.20	52.50	75.15	89.90
EP		8.60	28.55	66.10	84.15	94.60
LR	20,20,10	9.15	22.30	70.50	81.55	89.80
M		9.20	22.35	70.55	81.55	89.80
MB		9.20	22.30	70.45	81.50	89.80
CL		9.65	28.60	69.80	81.50	89.65
EP		7.60	23.50	63.40	78.25	88.75

Table 3.9: Empirical power (%) of the test statistics LR, M, MB, CL and EP; critical values based on 10,000 replications; power based on 2000 replications. L = 3, α = 0.10; k = 1.5.

		$(\lambda_1,\lambda_2,\lambda_3)$				
Tests	n_1,n_2,n_3	(.1,.12,.12)	(.1,.1,.15)	(.1,.2,.2)	(.1,.2,.25)	(.1,.2,.3)
LR	10,10,10	13.70	27.65	57.45	71.30	82.05
M		13.60	27.60	57.30	71.35	81.80
MB		13.60	27.60	57.20	71.20	81.80
CL		13.10	29.75	52.30	66.60	80.25
EP		14.00	27.60	60.80	74.65	85.45
LR	20,20,20	16.80	46.25	85.90	95.30	98.80
M		16.80	46.25	85.90	95.30	98.80
MB		16.80	46.25	85.90	95.25	98.80
CL		17.10	48.35	84.10	94.65	98.50
EP		17.00	45.80	87.30	95.90	98.80
LR	10,10,20	14.00	35.45	61.25	80.45	90.95
M		13.95	35.40	61.15	80.50	90.95
MB		13.95	35.40	61.05	80.40	90.85
CL		13.25	32.30	53.05	72.15	86.35
EP		15.00	33.40	66.95	82.75	92.90
LR	15,15,20	14.85	41.85	74.65	89.60	96.75
M		14.90	41.85	74.60	89.65	96.75
MB		14.95	41.80	74.45	89.70	96.75
CL		14.30	41.00	69.65	86.35	96.10
EP		16.05	41.95	77.95	91.45	97.65
LR	20,20,10	15.95	32.90	80.45	89.30	94.15
M		15.90	32.90	80.45	89.30	94.15
MB		15.85	32.75	80.35	89.25	94.15
CL		16.95	38.70	80.90	89.05	94.55
EP		14.25	35.50	77.30	86.80	94.10

Table 3.10: Empirical level (%) of the statistics CL;
$\lambda = 0.1$, $k = 1.5$, $L = 5, 10$.

	L = 5		
	α		
n_1,n_2,n_3,n_4,n_5	10.0	5.0	1.0
10,10,10,10,10	9.9	4.2	0.8
20,20,20,20,20	10.5	4.6	0.7
10,10,10,20,20	10.1	5.2	0.8
15,15,15,20,20	9.3	4.3	1.1
20,20,20,10,10	10.8	5.3	0.7

	L = 10		
	α		
n_1,n_2,n_3,n_4,n_5 n_6,n_7,n_8,n_9,n_{10}	10.0	5.0	1.0
10,10,10,10,10 10,10,10,10,10	9.6	4.8	1.0
20,20,20,20,20 20,20,20,20,20	9.6	4.5	0.7
10,10,10,10,10 10,20,20,20,20	9.3	4.3	0.8
15,15,15,15,15 15,20,20,20,20	10.3	5.3	1.2
20,20,20,20,20 20,10,10,10,10	10.7	5.6	1.6

Table 3.11: Empirical power (%) of the statistics CL and EP based on empirically calculated critical values; k = 1.5, L = 5, 10; α = 0.05.

Tests	n_1,n_2,n_3,n_4,n_5	$(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ (.1,.1,.15, .15,.15)	(.1,.15,.15 .2,.25)	(.1,.1,.2 .2,.3)	(.1,.2,.3 .4,.5)
		L = 5			
CL	10,10,10,10,10	20.4	47.3	81.2	93.4
EP		20.8	50.7	80.6	97.7
CL	20,20,20,20,20	42.2	84.7	99.3	100
EP		41.6	86.0	99.0	100
CL	10,10,10,20,20	17.4	53.1	86.8	95.8
EP		25.8	61.4	91.1	99.4
CL	15,15,15,20,20	31.0	73.9	96.4	99.7
EP		34.9	76.9	96.6	99.9
CL	20,20,20,10,10	40.1	72.0	96.9	99.7
EP		24.9	65.8	91.5	99.9

	n_1,n_2,n_3,n_4,n_5 n_6,n_7,n_8,n_9,n_{10}	$(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10})$ (.1,.1,.15, .15,.15,.15,.15, .15,.15,.15)	(.1,.1,.15, .15,.15,.2, .2,.2,.2,.2)	(.1,.15,.15 .2,.2,.2,.25, .25,.25,.25)	(.1,.1,.15 .2,.25,.25, .3,.3,.3,.3)
		L = 10			
CL	10,10,10,10,10	16.5	45.6	50.1	87.8
EP	10,10,10,10,10	17.7	47.3	58.3	91.6
CL	20,20,20,20,20	34.0	86.5	90.3	100
EP	20,20,20,20,20	38.7	84.3	93.3	100
CL	10,10,10,10,10	13.6	43.5	50.4	90.8
EP	10,20,20,20,20	21.2	53.0	62.7	95.6
CL	15,15,15,15,15	21.3	69.2	77.3	99.4
EP	15,20,20,20,20	30.0	71.5	83.9	99.6
CL	20,20,20,20,20	35.0	80.5	84.1	99.7
EP	20,10,10,10,10	18.4	59.9	76.0	98.0

Table 3.12. Empirical levels (%) of the test statistics LRk and CLk based on 2000 replications; k = 0.5, λ = (0.1,0.3).

Tests	n_1,n_2	L = 2	α 0.01	0.05	0.10
LRk	10,10		1.50	6.15	11.20
CLk			0.25	3.75	9.55
LRk	20,20		1.50	6.50	12.45
CLk			0.80	5.40	11.05
LRk	10,20		1.05	6.00	12.00
CLk			0.45	3.40	9.35
LRk	20,10		1.55	5.65	11.60
CLk			0.75	4.10	8.45
LRk	20,15		1.40	5.70	10.90
CLk			0.55	4.25	9.95
	n_1,n_2,n_3			L = 3	
LRk	10,10,10		1.35	7.15	13.15
CLk			0.90	4.45	9.50
LRk	20,20,20		1.28	6.20	11.64
CLk			0.92	4.75	10.24
LRk	10,10,20		1.52	6.72	13.48
CLk			1.00	4.92	9.32
LRk	20,20,10		1.35	6.40	12.50
CLk			0.80	4.50	9.10
LRk	20,15,15		1.25	5.45	10.35
CLk			0.55	4.30	8.80

Table 3.13: Empirical power (%) of the statistics LRk and CLk corresponding to nominal level $\alpha = 0.01$; critical values based on 10,000 replications; power based on 2000 replications. $L = 2$, $\lambda = (0.1, 0.3)$ and $L = 3$, $\lambda = (0.1, 0.2, 0.3)$.

Tests	n_1,n_2	(k_1,k_2) (.5,.5)	(.5,.8)	(.5,1)	(.5,1.2)	(.5,1.5)
LRk	10,10	0.80	3.70	7.25	12.70	21.35
CLk		0.90	3.70	7.25	12.65	20.75
LRk	20,20	0.75	8.30	19.05	33.35	54.15
CLk		0.70	8.00	18.85	32.40	53.30
LRk	10,20	0.90	4.35	10.90	19.30	34.15
CLk		1.30	11.95	23.80	35.40	51.80
LRk	20,10	1.05	4.70	10.55	16.65	28.20
CLk		0.90	0.35	0.95	1.50	2.50
LRK	20,15	1.05	8.25	18.90	29.20	48.85
CLk		0.90	5.10	13.65	21.55	40.10

Tests	n_1,n_2,n_3	(k_1,k_2,k_3) (.5,.6,.8)	(.5,.8,.1)	(.5,1,1.2)	(.5,1.2,1.5)	(.5,1.5,2)
LRk	10,1C,10	3.70	5.95	12.00	22.45	37.35
CLk		2.80	6.85	14.40	25.20	43.05
LRk	20,20,20	4.80	16.35	33.30	55.50	80.05
CLk		5.60	19.50	39.65	61.55	84.45
LRk	10,10,20	4.05	8.05	15.90	29.15	50.00
CLk		7.05	14.65	25.90	41.90	62.90
LRk	20,20,10	4.40	12.35	27.15	45.25	70.25
CLk		2.25	11.10	25.85	43.90	68.75
LRk	20,15,15	4.50	14.60	29.50	50.80	72.45
CLk		2.95	12.75	27.95	49.10	71.70

Table 3.14: Empirical power (%) of the statistics LRk and CLk corresponding to nominal level $\alpha = 0.05$; critical values based on 10,000 replications; power based on 2000 replications. $L = 2$, $\lambda = (0.1, 0.3)$ and $L = 3$, $\lambda = (0.1, 0.2, 0.3)$.

Tests	n_1, n_2	(k_1, k_2) (.5,.5)	(.5,.8)	(.5,1)	(.5,1.2)	(.5,1.5)
LRk	10,10	4.50	13.05	23.10	31.90	45.85
CLk		4.40	13.00	22.80	31.65	45.30
LRk	20,20	3.90	23.50	43.30	60.45	80.55
CLk		3.90	23.50	43.10	60.25	80.30
LRk	10,20	4.30	15.35	28.20	39.75	57.35
CLk		4.50	22.10	37.30	49.50	66.95
LRk	20,10	4.20	15.50	27.25	39.10	56.10
CLk		4.00	10.40	19.45	28.85	46.05
LRK	20,15	5.05	20.85	38.00	53.45	72.75
CLk		5.20	18.40	34.45	48.50	69.55

Tests	n_1, n_2, n_3	(k_1, k_2, k_3) (.5,.6,.8)	(.5,.8,.1)	(.5,1,1.2)	(.5,1.2,1.5)	(.5,1.5,2)
LRk	10,10,10	10.85	17.35	30.25	42.85	60.80
CLk		10.55	19.40	32.25	45.30	63.15
LRk	20,20,20	16.60	35.95	57.35	76.60	92.70
CLk		16.90	38.70	60.15	79.55	94.00
LRk	10,10,20	11.80	21.50	35.45	49.95	69.30
CLk		17.15	30.10	44.80	60.95	78.25
LRk	20,20,10	14.35	28.00	47.45	67.65	87.10
CLk		10.80	26.20	47.10	66.85	86.85
LRk	20,15,15	14.85	31.30	50.85	70.00	86.15
CLk		12.85	30.95	51.10	71.30	87.25

Table 3.15: Empirical power (%) of the statistics LRk and CLk corresponding to nominal level $\alpha = 0.10$; critical values based on 10,000 replications; power based on 2000 replications. $L = 2$, λ = (0.1,0.3) and $L = 3$, λ = (0.1,0.2,0.3).

		(k_1,k_2)				
Tests	n_1,n_2	(.5,.5)	(.5,.8)	(.5,1)	(.5,1.2)	(.5,1.5)
LRk	10,10	9.30	22.60	34.75	45.40	59.95
CLk		9.30	22.45	34.55	44.60	59.50
LRk	20,20	8.25	34.95	56.35	71.25	88.35
CLk		8.25	34.90	56.35	71.20	88.30
LRk	10,20	8.85	23.85	39.55	52.35	68.90
CLk		9.55	29.80	45.00	58.55	74.85
LRk	20,10	8.50	27.30	42.05	56.00	72.30
CLk		8.10	21.40	34.90	48.05	64.85
LRK	20,15	9.75	32.60	51.05	67.10	83.05
CLk		9.45	30.85	48.65	64.10	81.40

		(k_1,k_2,k_3)				
	n_1,n_2,n_3	(.5,.6,.8)	(.5,.8,.1)	(.5,1,1.2)	(.5,1.2,1.5)	(.5,1.5,2)
LRk	10,10,10	17.35	28.60	42.15	53.85	71.85
CLk		18.00	29.60	43.50	56.35	74.25
LRk	20,20,20	27.10	47.65	68.85	84.10	96.05
CLk		27.25	48.85	70.10	85.40	96.55
LRk	10,10,20	19.05	32.15	46.05	61.50	78.85
CLk		25.75	39.15	54.45	69.15	84.65
LRk	20,20,10	23.20	40.00	61.20	77.85	92.60
CLk		18.90	37.95	60.25	77.45	92.45
LRk	20,15,15	25.45	44.40	63.30	80.50	91.55
CLk		22.35	42.45	62.00	80.20	91.30

CHAPTER 4

TESTING HYPOTHESES IN MULTIPLE SAMPLES FROM TWO PARAMETER EXPONENTIAL DISTRIBUTIONS

4.1 INTRODUCTION

The two parameter exponential distribution is often proposed for modeling the lifetime distribution of items such as electronic components, light bulbs, etc. or the time to mortality. Recall the two parameter exponential distribution having probability density function

$$f(t;\ \mu, \theta) = \frac{1}{\theta} \exp\left(-\left(\frac{t-\mu}{\theta}\right) \right), \qquad t \geq \mu, \tag{4.1.1}$$

where the location parameter μ is interpreted as the minimum (or the guarantee) time before which no failures or deaths occur, and the scale parameter, as the mean life. When $\mu = 0$, the model in (4.1.1) reduces to the one parameter exponential distribution. Inference procedures for the parameters μ and θ have been dealt with, among others, by Lawless (1982). Chapter 6 of this thesis deals with confidence interval procedures for these parameters.

Often data arise in the form of multiple samples. When $\mu = 0$, one may be interested in comparing the means. If the means of the exponential distributions are equal, then reliabilities, percentiles, hazard rates and other quantities are equal. For this purpose, analytic methods such as likelihood ratio test, Bartlett test, for both complete and censored samples, have been proposed in the statistical literature (See Lawless, 1982; Nelson, 1982). When the location parameter μ is not equal to zero but is known, statistical analysis for the scale parameter θ can be carried out as for the one parameter exponential distribution since $(t-\mu)$ follows a one parameter exponential distribution. For studies involving L (≥ 2) groups from the populations having pdf (4.1.1), many authors have dealt with the problem of testing the equality of location parameters with or without the assumption of common scale parameters across all the populations, based on failure censored samples. In an attempt to compare equivalence of two two parameter exponential distributions Epstein and Tsao (1953) proposed several hypotheses and derived a likelihood ratio statistic for each case. For the same problem, Perng (1978) proposed a test statistic obtained by combining two independent test statistics, whose null distributions were based on the F distributions. Hsieh (1981) obtained a likelihood ratio test and approximated its distribution by a chi squared distribution. For $L \geq 2$, Hogg and Tanis (1963) described an iterative procedure based on likelihood ratio, for testing equality of the scale parameters and location parameters of several independent exponential distributions, which was essentially the repeated use of a procedure by Epstein and Tsao (1953). For the same problem, for $L \geq 3$, Singh and Narayan (1983) derived a likelihood ratio test using unequal sample sizes, and approximated its distribution by an F distribution. For testing the equality of L (≥ 2) location parameters with unspecified scale parameters Hsieh (1986) developed a modified likelihood ratio test procedure which was a generalization of Epstein and Tsao (1953). For the comparison of two exponential location parameters with the assumption of a common scale parameter, Kumar and Patel (1971) proposed a test based on ordered samples and derived the null distribution of the test statistic. In this case, Tiku (1981) developed an approximate procedure based on t distribution for doubly censored data. The power function of this statistic was given by Khatri (1981). Kambo and Awad (1985) generalized Tiku's statistic for testing the homogeneity of several location parameters assuming a common scale parameter across the populations. Considering Type II censored data, Singh (1983) proposed a likelihood ratio test for the same problem. However, very little attempt has been made to develop and evaluate statistics for testing the assumption of a common scale parameter. Note, testing the assumption of a common scale parameter is equivalent to testing the equality of mean life times of a number of exponential populations. In this case, Epstein and Tsao (1953) gave a likelihood ratio statistic, and Perng (1978) suggested a statistic based on F distribution for two samples. For $L \geq 2$, Singh (1985) described an approximate test, based on likelihood ratio, which was originally proposed by Bartlett (1937).

In this chapter we deal with testing the equality of scale parameters of L (≥ 2) groups from the two parameter exponential populations in the presence of unspecified location parameters based on complete and failure censored data. For this purpose, various estimation procedures for the scale parameters are considered in section 4.2. In section 4.3 we derive a likelihood ratio statistic (LR), a marginal likelihood ratio statistic (ML), a $C(\alpha)$ statistic (CM) (Neyman, 1959) based on the marginal likelihood estimate of the scale parameters under the null hypothesis and an extremal scale parameter ratio statistic (EP)(McCool, 1979). We show in the following section that the marginal likelihood ratio statistic (ML) is equivalent to the modified Bartlett test statistic discussed by Singh (1985), Lawless (1982) and Nelson (1982). Bartlett's small sample correction to the statistic ML is also taken into account and is called the modified marginal likelihood ratio statistic (MB). The performance of the test statistics LR, ML, MB, CM and EP are examined in terms of size and power by conducting Monte Carlo simulation studies and are discussed in section 4.4. Some examples are given in section 4.5.

4.2 ESTIMATION

Suppose $t_{i1} \le \dots \le t_{ir_i}$ (i = 1,...,L) is a set of the first r_i observations in a random sample of size n_i taken from the ith, i = 1,...,L, two parameter exponential population having pdf

$$f(t;\, \mu,\theta) = \frac{1}{\theta_i} \exp\left\{\frac{(t-\mu_i)}{\theta_i}\right\},\ t \ge \mu_i;\ \theta_i > 0\ .$$

The parameters μ_i and θ_i are the location and scale parameters respectively of the ith population. It is easily seen that when $r_i = n_i$, for all i, we deal with the complete samples. Our interest is to test

H_0: $\theta_1 = \dots = \theta_L$ ($=\theta$)

against

H_1: at least two θ's are unequal, in the presence of unspecified location parameters $\mu_1,\dots, \mu_L$.

4.2.1 Maximum Likelihood Esimation

Define $\mu = (\mu_1,\dots, \mu_L)'$ and $\theta = (\theta_1,\dots,\theta_L)'$.

When the data are Type II censored, the likelihood function $L(\mu, \theta)$, under the alternative hypothesis is

$$L(\mu, \theta) = \prod_{i=1}^{L} \left\{ \frac{n!}{(n_i - r_i)!} \left(\prod_{j=1}^{r_i} f(t_{ij}; \mu_i, \theta_i) \right) \left(S(t_{ir_i}) \right)^{(n_i - r_i)} \right\}.$$

Apart from a constant, the likelihood function $L(\mu_i, \theta_i)$ for the ith group from the two parameter exponential population is given by

$$L(\mu_i, \theta_i) = \frac{1}{\theta_i^{r_i}} \exp\left[-\frac{1}{\theta_i} \left(\sum_{j=1}^{r_i} (t_{ij} - \mu_i) + (n_i - r_i)(t_{ir_i} - \mu_i) \right) \right]. \tag{4.2.1}$$

The MLEs of the parameters μ_i and θ_i; $i = 1,\dots,L$, are easily obtained, but the usual method of equating the derivative $\partial l/\partial \mu_i$ to zero is not applicable, since the maximum occurs on a boundary. It may be noted that for the ith sample, the likelihood function increases with μ_i, but $\mu_i \le t_{i1} \le \dots \le t_{ir_i}$, $i = 1,\dots,L$. So, the maximum likelihood estimates of μ_i is $\tilde{\mu}_i = t_{i1}$, $i = 1,\dots,L$.

With $\tilde{\mu}_i = t_{i1}$, $i = 1,\dots,L$, the maximum likelihood estimate $\tilde{\theta}_i$, $i = 1,\dots,L$, is the θ_i value that maximizes the likelihood $L(\tilde{\mu}_i, \theta_i)$ or log of the likelihood $L(\tilde{\mu}_i, \theta_i)$. Alternatively, $\tilde{\theta}_i$, $i = 1,\dots,L$, can be obtained by solving the equation

$$\frac{\partial}{\partial \theta_i} \log L(\tilde{\mu}_i, \theta_i) = 0, \qquad i = 1,\dots,L.$$

Taking logarithms of $L(\tilde{\mu}_i, \theta_i)$, we obtain the log likelihood function $l(\tilde{\mu}_i, \theta_i)$, which is given by

$$l\,(\hat{\mu}_i,\theta_i) = \sum_{i=1}^{L}\left[-\,r_i \log \theta_i - \frac{1}{\theta_i}\left(\sum_{j=1}^{r_i} (t_{ij}-t_{i1}) + (n_i-r_i)\,(t_{ir_i}-t_{i1}\right)\right].$$

Define

$$S_i = \sum_{j=1}^{r_i} (t_{ij} - t_{i1}) + (n_i - r_i)\,(t_{ir_i} - t_{i1}).$$

Then

$$l(\hat{\mu}_i\,,\theta_i) = -\sum_{i=1}^{L} (\,r_i \log \theta_i + S_i/\theta_i\,)\,.$$

The maximum likelihood equation

$$\frac{\partial l}{\partial \theta_i} = S_i/\hat{\theta}_i^2 - r_i/\hat{\theta}_i = 0 \;\text{ implies }\; \hat{\theta}_i = S_i/r_i,\;\; i = 1,\ldots,L.$$

Under the null hypothesis H_0, the maximum likelihood estimate $\hat{\mu}_i$, $i = 1,\ldots,L$, remains unchanged; that is $\hat{\mu}_i = t_{i1}$, $i = 1,\ldots,L$. Now, the log likelihood function $l(\hat{\mu}, \theta)$ reduces to

$$l(\hat{\mu},\theta) = -\sum_{i=1}^{L} (r_i \log \theta + S_i/\theta) = R \log \theta + S/\theta,$$

where $S = \sum_{i=1}^{L} S_i$ *and* $R = \sum_{i=1}^{L} r_i$.

Thus, the estimating equation $\partial l/\partial\theta = -R/\hat{\theta} - S/\hat{\theta}^2 = 0$ implies $\hat{\theta} = S/R$, which is the maximum likelihood estimator of a common θ under the null hypothesis. Now, in section 2.14, $2\,S_i/\theta_i \sim \chi^2(2(r_i-1))$ and hence $E(S_i) = \theta_i\,(r_i-1)$, $i = 1,\ldots,L$. Then $E(\hat{\theta}_i) = (r_i-1)\theta_i/r_i \neq \theta_i$ and $E(\hat{\theta}) = (R-L)\theta/R \neq \theta$. It is evident that the MLEs $\hat{\theta}_i$, $i = 1,\ldots,L$ and $\hat{\theta}$ are biased estimates respectively for θ_i, $i = 1,\ldots,L$ and θ.

4.2.2 Marginal Likelihood Estimation

The marginal likelihood procedure discussed by Kalbfleisch and Sprott (1970) is applied to eliminate the nuisance parameters μ_i, $i = 1,\ldots,L$ from the likelihood function (4.2.1). Ignoring the constant term, the marginal likelihood for $\theta_1,\ldots,\theta_L$ is obtained as

$$L_m(\theta) = \prod_{i=1}^{L} \frac{1}{\theta_i^{(r_i-1)}} \exp\left(-\frac{S_i}{\theta_i} \right).$$

Taking logarithm of $L_m(\theta)$, we have

$$l_m(\theta) = -\sum_{i=1}^{L} \left[(r_i-1)\log\theta_i + \frac{S_i}{\theta_i} \right]. \tag{4.2.2}$$

Maximum marginal log likelihood equations are obtained by equating the partial derivatives of the log marginal likelihood function $l_m(\theta)$ to zero. Accordingly, we obtain, under H_1,

$$\frac{\partial l_m}{\partial\theta_i} = \left(\frac{S_i}{\theta_i^2} - \frac{(r_i-1)}{\theta_i} \right) = 0, \quad i = 1,\ldots,L,$$

and under H_0 the maximum likelihood equation is

$$\frac{\partial l_m}{\partial \theta} = \sum_{i=1}^{L} \left(\frac{S_i}{\theta^2} - \frac{(r_i-1)}{\theta} \right) = 0.$$

The maximum marginal likelihood estimate (MMLE) $\breve{\theta}_{im}$ of θ_i follows as $\breve{\theta}_{im} = S_i/(r_i-1)$, $i = 1,\ldots,L$ and MMLE $\hat{\theta}_m$ of common θ follows as $\hat{\theta}_m = S/(R-L)$. We can see that $E(\breve{\theta}_{im}) = \theta_i$, $i = 1,\ldots,L$ and $E(\hat{\theta}_m) = \theta$. Thus the maximum marginal likelihood estimates of θ_i, $i = 1,\ldots,L$ and θ are unbiased.

4.3 TEST STATISTICS

4.3.1 Likelihood Ratio Statistic (LR)

Suppose that $l(\bar{\mu},\bar{\theta})$ denotes the maximum value of the log likelihood function under the alternative hypothesis H_1 and $l(\hat{\mu},\hat{\theta})$ denotes the maximum value of the log likelihood under H_0. Then we have

$$l(\bar{\mu},\bar{\theta}) = -\sum_{i=1}^{L} \left(r_i \log \bar{\theta}_i + \frac{S_i}{\bar{\theta}_i} \right) \quad and \quad l(\hat{\mu},\hat{\theta}) = -\sum_{i=1}^{L} \left(r_i \log \hat{\theta} + \frac{S_i}{\hat{\theta}} \right),$$

where $\bar{\theta}_i$, $i = 1,\ldots,L$ and $\hat{\theta}$ are defined as in section 4.2. Thus the log likelihood ratio statistic (LR) is given by

$$LR = 2\ (\ l(\bar{\mu},\bar{\theta}) - l(\hat{\mu},\hat{\theta})) = 2 \sum_{i=1}^{L} \left[r_i \log \left(\frac{\hat{\theta}}{\bar{\theta}_i}\right) \right]. \tag{4.3.1}$$

Under the null hypothesis, the distribution of the statistic LR is approximately chi- square with (L-1) degrees of freedom.

4.3.2 Marginal Likelihood Ratio Statistic (ML)

As discussed in section (4.2), $l(\bar{\theta}_m)$ denotes the maximum value of the log marginal likelihood function under H_1 and $l(\hat{\theta}_m)$ denotes the maximum value of the log marginal likelihood under H_0. Thus, we have

$$l(\bar{\theta}_m) = -\sum_{i=1}^{L}\left[(r_i-1)\log\bar{\theta}_{im} + \frac{S_i}{\bar{\theta}_{im}}\right]$$

and

$$l(\hat{\theta}_m) = -\sum_{i=1}^{L}\left[(r_i-1)\log\hat{\theta}_m + \frac{S_i}{\hat{\theta}_m}\right],$$

where $\bar{\theta}_{im}$, $i = 1,...,L$ and $\hat{\theta}_m$ are as in section (4.2). The log marginal likelihood ratio statistic (ML) is given by

$$ML = 2\,[l(\bar{\theta}_m) - l(\hat{\theta}_m)] = 2\sum_{i=1}^{L}\left\{(r_i-1)\log\left(\frac{\hat{\theta}_m}{\bar{\theta}_{im}}\right)\right\}, \tag{4.3.2}$$

which is approximately distributed as chi- square with (L-1) degrees of freedom.

4.3.3 Modified Marginal Likelihood Ratio Statistic (MB)

Since $\hat{\theta}_m = S/(R-L)$ and $\bar{\theta}_{im} = S_i/(r_i-1)$, $i = 1,...,L$ the expression for the statistic ML can be rewritten as

$$ML = 2\left[(R-L)\log\left(\frac{S}{(R-L)}\right) - \sum_{i=1}^{L}(r_i-1)\log\left(\frac{S_i}{(r_i-1)}\right)\right].$$

Define $V_i = 2(r_i-1)$, $i = 1,...,L$ and $V = \sum_{i=1}^{L} V_i$. Then the statistic ML reduces to

$$ML = V \log(S/V) - \sum_{i=1}^{L} V_i \log(S_i/V_i).$$

From section 2.14, $2S_i/\theta_i \sim \chi^2(V_i)$ and it is easily seen that the statistic ML is modified Bartlett statistic (see Lawless, 1982; Singh, 1985). Now, using the Bartlett's small sample correction to the statistic ML, we obtain the statistic

$$MB = ML/C, \tag{4.3.3}$$

where

$$C = 1 + \frac{1}{3(L-1)}\left[\sum_{i=1}^{L} \frac{1}{V_i} - \frac{1}{V}\right],$$

which is also approximately distributed as chi- square with (L-1) degrees of freedom. Hsieh (1986) suggested this test statistic to test the homogeneity of several scale parameters from two parameter exponential populations.

4.3.4 Extremal Scale Parameter Ratio Statistic (EP)

As mentioned in section 4.1, Epstein and Tsao (1953) and Perng (1978) discussed the two sample problem of testing the two parameter exponential scale parameters by considering the ratio of the estimates of the scale parameters. We here extend this technique to test the homogeneity of L ($\geq$2) scale parameters. In terms of the estimates of the exponential scale parameters, the extremal scale parameter ratio statistic (EP) is given by

$$EP = \frac{\underset{1\le i\le L}{Max}\{\tilde{\theta}_i\}}{\underset{1\le i\le L}{Min}\{\tilde{\theta}_i\}} = \frac{\underset{1\le i\le L}{Max}\{\tilde{\theta}_{im}\}}{\underset{1\le i\le L}{Min}\{\tilde{\theta}_{im}\}}. \tag{4.3.4}$$

For $L = 2$ and $r_1 = r_2 = r$, the distribution of EP is truncated $F((2r-2),(2r-2))$ and the critical values can be obtained explicitly as discussed in chapter 3. Since $2 S_i/\theta_i \sim \chi^2(2r_i-2)$, for $L = 2$, under the null hypothesis H_0 the ratio

$F_t = [S_1(r_2-1)]/[S_2(r_1-1)] = \tilde{\theta}_{1m}/\tilde{\theta}_{2m} \sim F((2r_1-2),(2r_2-2))$.

An appropriate two sided test based on the statistic EP is to reject H_0 if either $F_t \le F_{(1-\alpha/2)}((2r_1-2),(2r_2-2))$ or $F_t \ge F_{\alpha/2}((2r_1-2),(2r_2-2))$, where $F_\alpha(v_1,v_2)$ denotes the upper αth quantile of the F distribution with degrees of freedom v_1 and v_2. For $L > 2$, the distribution of EP is not known and the percentage points need to be evaluated by Monte Carlo simulations.

4.3.5 C(α) Statistic (CM)

In this section, we derive a $C(\alpha)$ statistic from the marginal likelihood l_m given in section 4.2.2. Suppose that the alternative hypothesis is defined by $\theta_i = \theta + \phi_i$, $i = 1,\ldots,L$ with $\phi_L = 0$. Then testing the null hypothesis H_0 is equivalent to testing H_0: $\phi_i = 0$, $i = 1,\ldots,(L-1)$. After reparametrizing the scale parameters, the log marginal likelihood function l_m can be given as

$$l_i = -\sum_{i=1}^{L}\left[(r_i-1)\log(\theta+\phi_i) + \frac{S_i}{(\theta+\phi_i)}\right].$$

Define $\phi = (\phi_1,\ldots,\phi_{L-1})'$. We obtain, for $i = 1,\ldots,(L-1)$,

$$\psi_i = \left.\frac{\partial l_m}{\partial \phi_i}\right|_{\phi=0} = \frac{S_i}{\theta^2} - \frac{(r_i-1)}{\theta}, \quad and \quad \eta = \left.\frac{\partial l_m}{\partial \theta}\right|_{\phi=0} = \frac{S}{\theta^2} - \frac{(R-L)}{\theta}.$$

The variance- covariance of $\psi = (\psi_1,...,\psi_{L-1})'$ is obtained as $G - ab^{-1}a'$, where the (i,j)th (i,j = 1,...,(L-1)) element of G is

$$g_{ij} = E\left(- \left.\frac{\partial^2 l_m}{\partial\phi_i\partial\phi_j}\right|_{\phi=0}\right) = \begin{cases} \frac{(r_i-1)}{\theta^2}, & i=j \\ 0 \ , & i\neq j \end{cases} ,$$

$$a = (a_1,...,a_{(L-1)})'$$

with

$$a_i = E\left(- \left.\frac{\partial^2 l_m}{\partial\phi_i\partial\theta}\right|_{\phi=0}\right) = \frac{(r_i-1)}{\theta_i^2}$$

and

$$b = E\left(- \left.\frac{\partial^2 l_m}{\partial\theta^2}\right|_{\phi=0}\right) = \frac{(R-L)}{\theta^2}.$$

Replacing θ by $\hat{\theta}_m$, the maximum marginal likelihood estimator of θ which is root-n consistent and after some simplification we obtain the $C(\alpha)$ statistic CM as

$$CM = \psi' (G-ab^{-1}a')^{-1}\ \psi = \sum_{i=1}^{L} \left(\frac{S_i}{\hat{\theta}_m}\ (r_i-1)\right)^2 / (r_i-1) \ .$$

Since $\tilde{\theta}_{im} = S_i/(r_i-1)$, i = 1,...,L, the quantity CM can be written as

$$CM = \sum_{i=1}^{L} (r_i - 1) \left(\frac{\tilde{\theta}_{im} - \hat{\theta}_m}{\hat{\theta}_m} \right)^2 ,$$

which is approximately distributed as chi- square with (L-1) degrees of freedom.

4.4 SIMULATION STUDY

The performance of the statistics LR, ML, MB, EP and CM in terms of size and power, has been examined by conducting a simulation experiment. The samples from the two parameter exponential distribution were generated using IMSL (1987) subroutine RNEXP. Simulations were conducted for L = 2, 5 taking nominal levels $\alpha = 0.01$, 0.05 and 0.10 with various combinations of (n,r), where n and r represent, respectively, the sample size and the number of failures in the sample. Without loss of generality we chose $\mu = (\mu_1, \mu_2) = (0.0, 0.4)$ for L = 2 and $\mu = (\mu_1, \ldots, \mu_5) = (0.0, 0.2, 0.4, 0.6, 0.8)$ for L = 5 and the common true parameter $\theta = 1.0$. Each experiment, for computing empirical sizes, was based on 2,000 replications. The results are reported in Tables 4.1 and 4.2. For L = 2, the empirical level of the statistic EP was based on the distribution of the statistic $F_t = \tilde{\theta}_{1m}/\tilde{\theta}_{2m}$. Since the asymptotic distribution of the statistic EP is unknown when L > 2, it is not included in Table 4.2 for L = 5.

For the comparison of power performance of the all five statistics, a simulation study was conducted for L = 2, 5; $\alpha = 0.01$, 0.05, 0.10 and the same combinations of (n,r) presented in Tables 4.1 and 4.2. We computed the critical values from the empirical distribution of the foregoing statistics based on 10,000 replications. These critical values were then used in the power study. In this case, each experiment was based on 2,000 replications. The results of this study are reported in Tables 4.3 through 4.8.

Results

From Tables 4.1 and 4.2, it is evident that the statistic LR is often too liberal. For $L = 2$, the statistic CM holds the significance level reasonably well, although in some situations it is liberal for $\alpha = 0.10$ and conservative for $\alpha = 0.01$. As L increases (for example, for $L = 5$) this statistic shows some severe anti-conservative behaviour. The statistics ML and MB hold the nominal level well in all situations. For $L = 2$, the empirical size of the statistic EP based on F distribution is reasonably close to the nominal level. Note that two times the standard error of the probabilities reported is approximately 0.005, 0.010 and 0.013 respectively for $\alpha = 0.01$, 0.05 and 0.10.

From Tables 4.3 to 4.5, we can see that for $L = 2$, and $r_1 = r_2$, all five statistics LR, ML, MB, EP and CM have similar power even under heavy censoring. For $r_1 > r_2$ and $\theta_1 < \theta_2$, the statistic CM is most powerful and the power of the other statistics ML, MB and EP is closer to that of CM. For $r_1 < r_2$ and $\theta_1 < \theta_2$, the statistic LR is most powerful and the next best statistics are ML and MB. These behaviours are observed at all levels of α presented in Tables 4.3 to 4.5. In the case of $L = 5$, Tables 4.6 through 4.8 show that for $r_1 = ... = r_5$, the power of the statistic CM is always smaller than those of the other statistics. In the situations where θ_i's and r_i's are increasing, the statistics ML and MB provide better power and in the situations where decreasing r_i's are associated with increasing θ_i's, the statistic EP is the most powerful.

Note that the comparative performance of the statistics does not seem to depend on sample size configurations. Overall, the $C(\alpha)$ statistic shows anti- conservative behaviour and based on empirically computed critical values, this statistic does not show power advantage over other statistics. Based on empirically calculated percentage points the statistic EP is the most powerful in only one particular sample size and θ_i's values configuration. However, its null distribution is not known, so it is difficult to be used in practice. The performance of the statistics ML and MB are similar and they hold significance level well. However, the statistic ML is slightly liberal in very small sample size situations. Thus we recommend the statistic MB for use in practice in all situations based on its asymptotic null distribution as chi- squared with (L-1) degrees of freedom.

4.5 EXAMPLES

Example 1: The data in this example are from Perng (1978). Two processes for manufacturing a certain type of electronic components are to be used. Fifteen components from each process were on test and the number of failures in each process are defined as 12. The data are (in thousands of hours):

I. 0.044, 0.134, 0.142, 0.158, 0.216, 0.625, 0.649, 0.658, 1.062, 1.140, 1.159, 1.238

II. 0.060, 0.174, 0.237, 0.272, 0.335, 0.391, 0.670, 0.902, 1.543, 1.615, 2.013, 2.309.

The value of the statistic MB is 1.208 with p-value 0.272 which shows no evidence against the hypothesis of common scale parameters. The same conclusion is also reached by the other statistics, namely CM = 1.311 with p-value 0.252 and EP = 1.610 with p-value 0.277.

Example 2: The following example is taken from Singh (1985). The data are on intervals between failures (in hours) of the air-conditioning systems of a fleet of 13 Boeing 720 jet airplanes. In the following, we test the equality of three exponential scale parameters based on three random samples of sizes 27, 22, and 25. The first 15 ordered observations in each case are given below.

I. 1, 4, 11, 16, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80

II. 3, 5, 13, 14, 15, 22, 23, 30, 36, 39, 44, 46, 50, 72, 88

III. 10, 14, 20, 23, 24, 25, 26, 29, 44, 49, 56, 59, 60, 61, 62.

The value of the statistic MB is 1.827 with p-value = 0.401 which indicates very little evidence against the hypothesis of equality of the scale parameters. This conclusion is also reached by the other statistics CM = 2.104 with p-value = 0.349 and EP = 1.627 with p-value = 0.408.

Table 4.1: Empirical level (%) of the test statistics LR, ML, MB,EP and CM based on 2000 replications for both complete and Type II censored samples. For L = 2; μ = (0.0,0.4); common θ = 1.0.

		α		
Tests	n_1,r_1,n_2,r_2	10.0	5.0	1.0
LR	5,5,5,5	15.85	9.55	2.95
ML		14.40	6.40	1.15
MB		10.50	5.25	0.90
EP		10.50	5.30	0.90
CM		12.90	6.20	0.30
LR	10,10,10,10	11.35	5.30	1.45
ML		8.95	4.35	1.05
MB		8.05	4.05	1.00
EP		8.05	4.05	1.00
CM		9.75	4.40	0.80
LR	10,7,10,7	12.95	7.15	1.55
ML		10.10	5.35	0.65
MB		9.55	4.95	0.45
EP		9.55	4.95	0.45
CM		10.80	5.35	0.35
LR	10,5,10,5	14.50	8.65	2.40
ML		10.25	6.00	1.30
MB		9.55	5.25	1.15
EP		9.55	5.25	1.15
CM		11.75	5.90	0.45
LR	20,20,20,20	10.75	5.05	1.15
ML		9.70	4.25	0.95
MB		9.35	4.15	0.85
EP		9.35	4.15	0.85
CM		9.85	4.25	0.85
LR	20,15,20,15	10.45	5.30	1.00
ML		9.75	4.65	0.75
MB		9.45	4.40	0.65
EP		9.45	4.40	0.65
CM		9.90	4.65	0.55
LR	20,10,20,10	11.75	6.25	1.20
ML		10.00	4.85	0.75
MB		9.25	4.40	0.70
EP		9.25	4.40	0.70
CM		10.55	4.85	0.50
LR	20,5,20,5	13.80	7.65	2.10
ML		9.15	5.00	1.05
MB		8.40	4.55	1.00
EP		8.40	4.55	1.00
CM		10.80	4.90	0.30

Table 4.1 continued

LR	10,10,20,20	12.75	6.30	1.25
ML		9.95	4.95	0.80
MB		9.55	4.70	0.75
EP		9.85	4.60	0.60
CM		10.05	4.35	0.65
LR	10,7,20,15	12.50	7.35	1.50
ML		10.65	5.15	0.85
MB		9.95	4.60	0.70
EP		9.60	4.85	0.75
CM		10.75	4.40	0.30
LR	10,5,20,10	13.85	8.45	2.15
ML		9.95	4.70	1.00
MB		8.80	4.40	0.80
EP		9.50	4.40	0.90
CM		10.20	3.85	0.20
LR	20,20,10,10	10.65	5.45	1.05
ML		8.45	4.35	0.65
MB		8.25	4.00	0.65
EP		8.50	3.95	0.75
CM		8.75	4.15	0.35
LR	20,15,10,7	13.30	6.90	1.75
ML		10.20	4.80	0.85
MB		9.80	4.55	0.80
EP		9.55	4.45	0.80
CM		10.30	4.45	0.55
LR	20,10,10,5	13.60	8.10	2.20
ML		10.25	5.05	0.80
MB		9.60	4.70	0.70
EP		9.50	4.60	0.65
CM		10.40	4.50	0.75
LR	20,10,10,10	11.50	5.45	1.20
ML		9.65	4.50	0.85
MB		9.05	4.30	0.75
EP		9.10	4.30	0.75
CM		10.30	4.50	0.50
LR	30,10,20,10	12.40	6.45	1.35
ML		10.35	5.45	0.90
MB		10.00	5.25	0.80
EP		10.00	5.25	0.80
CM		11.10	5.45	0.55
LR	30,15,20,10	11.70	5.70	1.05
ML		9.55	4.75	0.85
MB		9.10	4.55	0.80
EP		9.45	4.40	0.75
CM		9.85	4.60	0.45

Table 4.2: Empirical level (%) of the test statistics LR, ML, MB and CM based on 2000 replications for both complete and Type II censored samples. For L = 5; μ = (0.0,0.2,0.4,0.6,0.8); common θ = 1.0.

Tests	n_1,r_1,n_2,r_2,n_3,r_3 n_4,r_4,n_5,r_5	α 10.0	5.0	1.0
LR	5,5,5,5,5,5	20.45	12.30	4.00
ML	5,5,5,5	11.60	6.30	1.30
MB		10.35	5.25	0.90
CM		14.00	7.95	2.45
LR	10,10,10,10,10,10	12.00	6.35	1.60
ML	10,10,10,10	8.90	4.55	0.90
MB		8.30	4.30	0.85
CM		10.65	5.80	1.20
LR	10,7,10,7,10,7	15.15	8.60	2.00
ML	10,7,10,7	9.75	4.65	0.85
MB		8.90	4.25	0.75
CM		11.15	5.80	1.65
LR	10,5,10,5,10,5	20.10	11.25	3.35
ML	10,5,10,5	10.40	5.00	0.95
MB		9.10	3.95	0.90
CM		14.10	6.95	2.10
LR	20,20,20,20,20,20	10.90	6.15	1.35
ML	20,20,20,20	9.25	5.00	0.90
MB		9.20	4.95	0.80
CM		10.50	5.70	1.10
LR	20,15,20,15,20,15	12.45	6.15	1.15
ML	20,15,20,15	10.40	4.80	0.90
MB		9.80	4.60	0.90
CM		11.65	5.70	1.15
LR	20,10,20,10,20,10	12.55	7.10	1.90
ML	20,10,20,10	9.40	4.90	1.10
MB		8.65	4.30	0.95
CM		10.50	5.30	1.30
LR	20,5,20,5,20,5	19.20	11.95	3.45
ML	20,5,20,5	11.20	5.75	1.35
MB		9.80	4.85	1.15
CM		14.85	7.85	1.90
LR	10,10,10,10,10,10	13.75	6.80	1.95
ML	20,20,20,20	10.20	5.10	1.50
MB		9.65	4.70	1.40
CM		11.95	5.90	1.50

Table 4.2 continued

LR	10,7,10,7,10,7	15.35	8.10	2.20
ML	20,15,20,15	9.60	5.30	1.25
MB		8.90	4.85	1.20
CM		11.20	5.55	1.55
LR	10,5,10,5,10,5	18.25	11.00	3.35
ML	20,10,20,10	10.25	5.50	1.20
MB		9.05	4.70	1.05
CM		12.75	6.50	1.75
LR	20,20,20,20,20,20	12.65	7.20	1.45
ML	10,10,10,10	9.80	5.40	0.90
MB		9.45	5.00	0.85
CM		10.60	5.50	1.15
LR	20,15,20,15,20,15	14.15	8.15	2.20
ML	10,7,10,7	10.15	5.20	0.90
MB		9.65	4.85	0.75
CM		11.10	5.55	1.00
LR	20,10,20,10,20,10	18.10	9.85	2.55
ML	10,5,10,5	10.10	5.35	0.95
MB		9.10	4.65	0.85
CM		11.15	6.20	1.30
LR	20,10,20,10,20,10	13.60	7.30	1.45
ML	10,10,10,10	9.45	4.85	0.85
MB		8.95	4.40	0.85
CM		11.10	5.85	1.20
LR	30,10,25,10,20,10	13.50	7.65	1.80
ML	15,10,10,10	9.80	5.30	0.90
MB		9.05	4.80	0.75
CM		12.10	6.10	1.50
LR	30,20,25,20,20,15	15.75	9.00	2.10
ML	15,10,10,5	10.35	4.75	0.65
MB		9.75	4.50	0.55
CM		11.00	5.35	1.10

Table 4.3: Empirical power (%) of the statistics LR, ML, MB, EP and CM corresponding to nominal size $\alpha = 0.01$; critical values based on 10,000 replications; power based on 2000 replications. L = 2.

Tests	n_1,r_1,n_2,r_2	(θ_1,θ_2) (1,1)	(1,1.5)	(1,2.0)	(1,2.5)
LR	5,5,5,5	1.00	1.95	4.45	7.45
ML		1.00	1.95	4.45	7.45
MB		1.00	1.95	4.45	7.45
EP		0.95	1.75	4.30	6.85
CM		1.00	1.95	4.45	7.45
LR	10,10,10,10	1.05	3.75	11.65	23.20
ML		1.05	3.75	11.65	23.20
MB		1.05	3.75	11.65	23.20
EP		1.05	3.15	10.55	20.55
CM		1.05	3.75	11.65	23.20
LR	10,7,10,7	0.65	3.05	7.45	13.25
ML		0.65	3.05	7.45	13.25
MB		0.65	3.05	7.45	13.25
EP		0.75	3.00	7.15	12.70
CM		0.65	3.05	7.45	13.25
LR	10,5,10,5	1.15	1.90	3.90	7.05
ML		1.15	1.90	3.90	7.05
MB		1.15	1.90	3.90	7.05
EP		1.15	1.90	3.90	7.05
CM		1.15	1.90	3.90	7.05
LR	20,20,20,20	1.10	9.45	32.80	61.00
ML		1.10	9.45	32.80	61.00
MB		1.10	9.45	32.80	61.00
EP		1.00	8.25	29.80	58.25
CM		1.10	9.45	32.80	61.05
LR	20,15,20,15	0.80	5.85	20.95	43.30
ML		0.85	5.85	20.95	43.30
MB		0.85	5.85	20.95	43.30
EP		0.85	6.35	22.80	45.00
CM		0.80	5.85	20.95	43.30
LR	20,10,20,10	0.80	3.75	12.05	24.65
ML		0.85	3.75	12.05	24.65
MB		0.85	3.75	12.05	24.65
EP		0.80	3.80	12.05	24.75
CM		0.85	3.75	12.05	24.65
LR	20,5,20,5	1.00	2.05	4.35	7.20
ML		1.00	2.05	4.35	7.20
MB		1.00	2.05	4.35	7.20
EP		1.05	2.15	4.50	7.35
CM		1.00	2.05	4.35	7.20

Table 4.3 continued

LR	10,10,20,20	0.70	6.10	17.70	36.45
ML		0.95	5.25	15.25	32.70
MB		0.95	5.25	15.25	32.70
EP		1.00	4.25	14.00	29.50
CM		0.95	3.45	12.55	26.00
LR	10,7,20,15	0.95	4.25	10.45	19.80
ML		0.85	3.55	8.90	17.35
MB		0.85	3.55	8.90	17.35
EP		0.80	2.85	7.45	14.95
CM		0.85	1.25	4.40	8.85
LR	10,5,20,10	0.85	2.70	6.30	10.70
ML		0.75	2.25	5.00	9.30
MB		0.75	2.25	5.00	9.30
EP		0.65	0.75	3.55	7.40
CM		0.70	0.50	1.55	2.65
LR	20,20,10,10	0.80	3.75	16.75	34.25
ML		0.80	5.80	21.45	41.90
MB		0.80	5.80	21.45	41.90
EP		0.70	6.35	22.20	43.20
CM		0.60	7.60	25.15	46.80
LR	20,15,10,7	0.90	2.25	8.90	18.30
ML		0.80	4.10	12.50	26.50
MB		0.80	4.10	12.50	26.20
EP		0.80	4.95	14.25	29.55
CM		1.00	6.50	18.10	34.35
LR	20,10,10,5	0.65	1.50	4.35	9.35
ML		0.70	2.90	7.70	14.75
MB		0.70	2.90	7.70	14.75
EP		0.80	3.40	9.20	17.30
CM		1.10	5.40	13.05	22.95
LR	20,10,10,10	0.90	3.90	13.05	25.50
ML		0.90	3.90	13.05	25.50
MB		0.90	3.90	13.05	25.50
EP		1.00	3.75	12.60	24.70
CM		0.90	3.90	13.05	25.50
LR	30,10,20,10	0.75	3.90	11.00	23.00
ML		0.75	3.90	11.00	23.00
MB		0.75	3.90	11.00	23.00
EP		0.75	4.10	11.35	23.70
CM		0.80	3.90	11.00	23.00
LR	30,15,20,10	0.90	4.25	15.50	31.75
ML		0.85	4.75	17.50	34.50
MB		0.85	4.75	17.50	34.50
EP		1.00	6.10	19.80	37.35
CM		1.00	6.70	21.75	39.30

Table 4.4: Empirical power (%) of the statistics LR, ML, MB, EP and CM corresponding to nominal size $\alpha = 0.05$; critical values based on 10,000 replications; power based on 2000 replications. L = 2.

		(θ_1,θ_2)			
Tests	n_1,r_1,n_2,r_2	(1,1)	(1,1.5)	(1,2.0)	(1,2.5)
LR	5,5,5,5	5.05	7.95	14.50	21.45
ML		5.00	7.95	14.50	21.45
MB		5.00	7.95	14.50	21.45
EP		5.05	7.95	14.50	21.45
CM		5.05	7.95	14.50	21.45
LR	10,10,10,10	4.45	12.90	29.70	48.85
ML		4.40	12.90	29.70	48.85
MB		4.40	12.90	29.70	48.80
EP		4.45	12.85	29.45	48.65
CM		4.45	12.90	29.70	48.85
LR	10,7,10,7	5.15	9.60	20.20	33.45
ML		5.15	9.60	20.20	33.45
MB		5.15	9.60	20.20	33.45
EP		5.10	9.55	19.95	33.05
CM		5.15	9.60	20.20	33.45
LR	10,5,10,5	5.20	8.25	13.45	20.90
ML		5.20	8.25	13.45	20.90
MB		5.20	8.25	13.50	20.90
EP		5.20	8.10	13.30	20.65
CM		5.20	8.25	13.45	20.90
LR	20,20,20,20	4.15	22.90	57.05	79.55
ML		4.15	22.90	57.05	79.55
MB		4.15	22.90	57.05	79.55
EP		4.10	22.70	57.00	79.30
CM		4.15	22.90	57.05	79.55
LR	20,15,20,15	4.40	16.75	43.90	66.85
ML		4.40	16.75	43.90	66.85
MB		4.40	16.75	43.90	66.85
EP		4.45	17.00	44.25	67.35
CM		4.40	16.75	43.90	66.85
LR	20,10,20,10	4.70	12.20	29.00	46.35
ML		4.70	12.20	29.00	46.35
MB		4.70	12.20	29.00	46.35
EP		4.60	12.00	28.60	46.10
CM		4.70	12.20	29.00	46.35
LR	20,5,20,5	4.85	7.75	15.15	22.80
ML		4.85	7.80	15.15	22.80
MB		4.85	7.75	15.15	22.80
EP		4.80	7.70	14.90	22.45
CM		4.85	7.75	15.15	22.80

Table 4.4 continued

LR	10,10,20,20	4.70	18.25	44.00	65.05
ML		5.00	15.10	39.10	60.45
MB		5.00	15.10	39.10	60.45
EP		4.80	14.05	36.25	56.85
CM		4.80	14.35	36.70	58.10
LR	10,7,20,15	4.70	13.15	28.70	46.00
ML		4.50	10.90	24.25	41.25
MB		4.50	10.90	24.25	41.25
EP		4.40	9.35	21.35	36.90
CM		4.45	9.40	21.45	37.15
LR	10,5,20,10	4.35	11.35	20.55	31.90
ML		4.50	9.55	17.10	27.10
MB		4.50	9.55	17.10	27.10
EP		4.30	8.15	14.60	22.80
CM		4.35	8.40	14.95	23.80
LR	20,20,10,10	4.40	13.10	35.00	57.40
ML		4.35	16.60	41.90	62.15
MB		4.35	16.60	41.90	62.15
EP		4.25	17.35	43.35	63.45
CM		4.30	17.55	43.65	63.85
LR	20,15,10,7	4.90	9.85	23.40	39.80
ML		4.55	12.55	30.10	47.50
MB		4.55	12.55	30.10	47.50
EP		4.65	13.55	31.85	49.75
CM		4.50	13.60	32.25	50.15
LR	20,10,10,5	4.85	7.00	14.60	24.75
ML		4.90	10.15	21.75	34.00
MB		4.90	10.15	21.75	34.00
EP		4.90	10.90	22.90	36.00
CM		4.95	11.40	23.90	36.85
LR	20,10,10,10	4.55	13.75	30.30	47.25
ML		4.55	13.75	30.30	47.25
MB		4.55	13.75	30.30	47.25
EP		4.50	13.95	30.40	47.60
CM		4.55	13.75	30.30	47.25
LR	30,10,20,10	5.55	13.60	31.05	49.60
ML		5.55	13.60	31.05	49.60
MB		5.55	13.60	31.05	49.60
EP		5.50	14.15	31.55	50.40
CM		5.55	13.60	31.05	49.60
LR	30,15,20,10	4.60	13.45	33.40	52.70
ML		4.70	15.05	36.35	55.70
MB		4.70	15.05	36.35	55.70
EP		4.60	16.35	37.95	57.15
CM		4.55	16.10	37.60	56.95

Table 4.5: Empirical power (%) of the statistics LR, ML, MB, EP and CM corresponding to nominal size $\alpha = 0.10$; critical values based on 10,000 replications; power based on 2000 replications. L = 2.

		(θ_1,θ_2)			
Tests	n_1,r_1,n_2,r_2	(1,1)	(1,1.5)	(1,2.0)	(1,2.5)
LR	5,5,5,5	10.00	14.05	22.45	32.30
ML		10.00	14.05	22.45	32.30
MB		10.00	14.05	22.45	32.30
EP		9.50	14.05	22.35	32.15
CM		10.00	14.05	22.45	32.30
LR	10,10,10,10	9.55	21.10	43.25	61.15
ML		9.50	21.10	43.25	61.15
MB		9.55	21.10	43.25	61.15
EP		9.60	21.05	43.20	61.15
CM		9.50	21.10	43.25	61.15
LR	10,7,10,7	9.90	16.80	32.05	46.80
ML		9.85	16.80	32.05	46.80
MB		9.90	16.80	32.05	46.80
EP		9.85	16.55	31.50	46.45
CM		9.85	16.80	32.05	46.80
LR	10,5,10,5	9.70	13.85	23.20	34.65
ML		9.70	13.85	23.20	34.65
MB		9.70	13.90	23.20	34.65
EP		9.75	13.60	22.95	34.05
CM		9.70	13.90	23.20	34.65
LR	20,20,20,20	9.55	33.30	68.75	88.40
ML		9.55	33.30	68.75	88.40
MB		9.55	33.30	68.75	88.40
EP		9.40	33.15	68.50	88.30
CM		9.55	33.30	68.75	88.40
LR	20,15,20,15	9.40	27.00	57.20	77.90
ML		9.40	27.00	57.20	77.90
MB		9.40	27.00	57.20	77.90
EP		9.45	27.10	57.30	78.10
CM		9.40	27.00	57.20	77.90
LR	20,10,20,10	9.95	20.95	41.00	59.90
ML		9.95	20.95	41.00	59.90
MB		9.95	20.95	41.00	59.90
EP		10.15	20.85	40.70	59.60
CM		9.95	21.00	41.00	59.90
LR	20,5,20,5	8.90	14.55	23.95	34.55
ML		8.90	14.55	23.95	34.55
MB		8.90	14.55	23.95	34.55
EP		9.05	14.05	23.50	33.90
CM		8.90	14.55	23.95	34.55

Table 4.5 continued

LR	10,10,20,20	10.05	28.85	55.95	75.70
ML		9.65	24.80	52.35	71.75
MB		9.65	24.80	52.35	71.75
EP		9.50	22.30	49.25	69.65
CM		9.70	25.10	52.45	72.25
LR	10,7,20,15	9.40	22.50	43.35	61.60
ML		9.55	18.50	37.95	55.00
MB		9.55	18.50	37.95	55.00
EP		9.70	17.10	33.90	50.90
CM		9.55	18.50	37.95	55.00
LR	10,5,20,10	9.65	18.50	32.95	47.10
ML		9.20	15.35	27.30	40.10
MB		9.20	15.35	27.30	40.10
EP		8.65	13.30	23.60	35.55
CM		9.15	15.35	27.25	40.00
LR	20,20,10,10	9.15	22.20	47.50	67.85
ML		8.75	26.80	53.30	73.35
MB		8.75	26.80	53.30	73.35
EP		8.85	27.40	54.00	73.80
CM		8.65	26.00	52.80	72.80
LR	20,15,10,7	9.80	15.65	34.65	52.20
ML		9.90	20.80	41.85	59.05
MB		9.90	20.80	41.85	59.05
EP		10.00	22.15	42.95	60.35
CM		9.95	20.65	41.55	59.00
LR	20,10,10,5	9.55	12.45	23.25	35.90
ML		9.80	17.45	31.95	44.45
MB		9.80	17.45	31.95	44.45
EP		9.95	19.55	34.40	47.80
CM		9.80	17.50	31.95	44.55
LR	20,10,10,10	9.50	22.70	42.00	61.90
ML		9.45	22.70	42.00	61.85
MB		9.50	22.70	42.00	61.90
EP		9.50	22.25	41.25	61.15
CM		9.50	22.70	42.00	61.90
LR	30,10,20,10	10.25	22.00	43.45	60.75
ML		10.25	22.00	43.45	60.75
MB		10.25	22.00	43.45	60.75
EP		10.25	22.10	43.60	60.75
CM		10.25	22.00	43.45	60.75
LR	30,15,20,10	9.60	22.25	45.25	64.20
ML		9.30	24.95	48.40	66.95
MB		9.35	24.95	48.40	66.95
EP		9.10	25.30	49.00	67.80
CM		9.30	24.95	48.15	66.85

Table 4.6: Empirical power (%) of the statistics LR, ML, MB, EP and CM corresponding to nominal size $\alpha = 0.01$; critical values based on 10,000 replications; power based on 2000 replications; L = 5.

Tests	n_1,r_1,n_2,r_2,n_3,r_3 n_4,r_4,n_5,r_5	$(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ (1,1.2,1.4,1.6,1.8)	(1,1.6,1.8,2,2.2)	(1.2,2.2,2.4,2.6)	(1.2.4,2.6,2.8,3)
LR	5,5,5,5,5,5	2.45	3.40	4.45	5.80
ML	5,5,5,5	2.45	3.40	4.45	5.80
MB		2.45	3.40	4.45	5.80
EP		2.30	3.15	4.40	5.85
CM		2.00	2.45	3.05	3.70
LR	10,10,10,10,10,10	4.75	8.40	13.80	20.85
ML	10,10,10,10	4.75	8.40	13.80	20.85
MB		4.75	8.40	13.80	20.85
EP		4.05	8.45	15.25	25.40
CM		4.60	6.10	7.80	9.65
LR	10,7,10,7,10,7	2.85	4.45	6.95	9.95
ML	10,7,10,7	2.85	4.45	6.95	9.95
MB		2.85	4.45	6.95	9.95
EP		2.70	4.85	7.80	12.80
CM		3.20	3.80	4.35	5.10
LR	10,5,10,5,10,5	1.95	2.55	3.80	5.05
ML	10,5,10,5	1.95	2.55	3.80	5.05
MB		1.95	2.55	3.80	5.05
EP		1.55	2.85	4.30	6.15
CM		2.05	2.50	3.10	3.55
LR	20,20,20,20,20,20	13.85	28.20	48.35	69.45
ML	20,20,20,20	13.85	28.20	48.35	69.45
MB		13.85	28.20	48.35	69.45
EP		12.10	29.15	53.15	74.60
CM		13.15	18.75	29.80	43.45
LR	20,15,20,15,20,15	8.90	18.45	31.75	48.00
ML	20,15,20,15	8.90	18.45	31.75	48.00
MB		8.90	18.45	31.75	48.00
EP		7.80	17.70	34.70	52.80
CM		7.40	10.15	14.40	19.60
LR	20,10,20,10,20,10	4.70	8.45	15.45	23.65
ML	20,10,20,10	4.70	8.45	15.45	23.65
MB		4.70	8.45	15.45	23.65
EP		4.05	8.10	16.45	23.65
CM		4.85	6.25	7.85	9.60
LR	20,5,20,5,20,5	2.00	2.50	3.40	4.90
ML	20,5,20,5	2.00	2.50	3.40	4.90
MB		2.00	2.50	3.40	4.90
EP		2.20	3.15	3.90	5.65
CM		2.45	2.45	2.80	3.20

Table 4.6 continued

LR	10,10,10,10,10,10	8.50	13.30	20.75	29.15
ML	20,20,20,20	6.55	10.90	17.10	25.35
MB		6.55	10.90	17.10	25.35
EP		6.05	12.30	22.10	33.60
CM		5.50	6.40	7.75	9.60
LR	10,7,10,7,10,7	5.40	8.30	11.35	15.85
ML	20,15,20,15	3.85	5.65	9.10	12.45
MB		3.85	5.55	9.10	12.50
EP		3.55	7.00	11.60	17.55
CM		3.00	3.10	3.80	4.35
LR	10,5,10,5,10,5	4.30	5.45	7.40	9.45
ML	20,10,20,10	2.65	3.85	5.05	6.65
MB		2.65	3.85	5.05	6.65
EP		2.85	4.35	6.75	10.10
CM		1.40	1.45	1.70	1.80
LR	20,20,20,20,20,20	6.80	18.50	38.60	60.15
ML	10,10,10,10	8.60	21.65	43.50	64.75
MB		8.60	21.65	43.50	64.75
EP		3.10	6.70	16.80	33.40
CM		10.95	18.70	29.40	42.50
LR	20,15,20,15,20,15	3.00	7.55	17.40	30.50
ML	10,7,10,7	5.00	11.80	22.20	38.45
MB		5.00	11.80	22.20	38.45
EP		0.85	1.60	4.15	8.20
CM		6.85	10.30	14.95	22.25
LR	20,10,20,10,20,10	2.10	4.65	8.65	13.80
ML	10,5,10,5	3.55	6.80	12.65	19.85
MB		3.55	6.80	12.65	19.85
EP		0.65	1.00	1.60	2.85
CM		4.20	5.55	7.65	9.65
LR	20,10,20,10,20,10	5.45	9.40	15.25	22.60
ML	10,10,10,10	5.45	9.40	15.25	22.60
MB		5.45	9.40	15.25	22.60
EP		4.30	8.75	16.45	27.00
CM		5.75	7.25	8.90	10.70
LR	30,10,25,10,20,10	5.45	8.45	14.50	22.35
ML	15,10,10,10	5.45	8.45	14.50	22.35
MB		5.45	8.45	14.50	22.35
EP		4.15	7.95	15.70	24.15
CM		4.85	6.10	7.30	9.15
LR	30,20,25,20,20,15	3.50	9.70	24.30	43.70
ML	15,10,10,5	6.25	16.70	33.75	54.70
MB		6.25	16.70	33.75	54.70
EP		0.45	0.55	0.80	1.55
CM		6.90	11.75	21.10	32.85

Table 4.7: Empirical power (%) of the statistics LR, ML, MB, EP and CM corresponding to nominal size $\alpha = 0.05$; critical values based on 10,000 replications; power based on 2000 replications; L = 5.

		$(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$			
Tests	n_1,r_1,n_2,r_2,n_3,r_3 n_4,r_4,n_5,r_5	(1,1.2,1.4,1.6,1.8)	(1,1.6,1.8,2.2,2.2)	(1,2.2,2.2,2.4,2.6)	(1,2.4,2.6,2.8,3)
LR	5,5,5,5,5,5	8.75	10.70	14.25	17.50
ML	5,5,5,5	8.75	10.70	14.25	17.50
MB		8.75	10.70	14.25	17.50
EP		7.80	9.90	13.85	18.30
CM		9.00	9.80	11.05	12.40
LR	10,10,10,10,10,10	14.50	21.90	32.70	45.25
ML	10,10,10,10	14.50	21.90	32.70	45.30
MB		14.50	21.90	32.70	45.25
EP		13.70	23.00	36.25	49.15
CM		14.15	17.95	23.60	29.30
LR	10,7,10,7,10,7	11.90	17.15	23.20	30.00
ML	10,7,10,7	11.90	17.15	23.20	30.00
MB		11.90	17.15	23.20	30.00
EP		10.65	16.75	23.65	31.50
CM		10.95	13.35	16.30	19.65
LR	10,5,10,5,10,5	8.70	11.45	15.50	19.00
ML	10,5,10,5	8.70	11.45	15.50	19.00
MB		8.70	11.45	15.50	19.00
EP		8.40	11.35	14.80	19.95
CM		8.25	9.55	11.05	12.30
LR	20,20,20,20,20,20	32.00	52.90	74.10	88.55
ML	20,20,20,20	32.00	52.90	74.10	88.55
MB		32.00	52.90	74.30	88.55
EP		30.15	53.75	77.55	90.55
CM		30.15	45.60	62.60	77.70
LR	20,15,20,15,20,15	23.15	38.55	57.70	73.70
ML	20,15,20,15	23.20	38.55	57.70	73.70
MB		23.20	38.55	57.70	73.70
EP		22.10	40.75	60.75	77.15
CM		23.05	31.95	44.55	57.15
LR	20,10,20,10,20,10	16.50	25.45	36.60	49.15
ML	20,10,20,10	16.50	25.45	36.60	49.15
MB		16.50	25.45	36.60	49.15
EP		14.05	23.80	36.80	50.50
CM		16.00	20.50	26.60	33.30
LR	20,5,20,5,20,5	9.10	11.50	15.45	19.20
ML	20,5,20,5	9.10	11.50	15.45	19.20
MB		9.10	11.50	15.45	19.20
EP		8.05	10.50	14.40	19.70
CM		9.05	11.10	12.55	14.45

Table 4.7 continued

LR	10,10,10,10,10,10	24.80	34.40	46.40	58.00
ML	20,20,20,20	20.40	29.80	41.05	55.55
MB		20.40	29.80	41.05	55.55
EP		19.90	32.60	48.05	63.60
CM		18.05	22.90	28.65	34.00
LR	10,7,10,7,10,7	17.45	23.70	31.65	40.10
ML	20,15,20,15	14.35	20.15	26.95	34.60
MB		14.35	20.15	26.95	34.60
EP		14.65	21.55	31.70	43.30
CM		13.05	15.05	17.45	20.15
LR	10,5,10,5,10,5	13.50	17.50	22.00	27.05
ML	20,10,20,10	10.60	12.65	17.65	22.35
MB		10.60	12.65	17.65	22.35
EP		11.30	16.05	21.05	28.30
CM		10.60	11.60	12.95	14.65
LR	20,20,20,20,20,20	20.40	40.45	64.75	81.40
ML	10,10,10,10	24.25	47.20	69.00	84.90
MB		24.25	47.20	69.00	84.90
EP		12.25	26.15	48.40	68.95
CM		25.20	40.50	59.35	75.25
LR	20,15,20,15,20,15	13.05	24.60	43.40	61.35
ML	10,7,10,7	16.80	31.70	51.15	67.55
MB		16.80	31.70	51.15	67.55
EP		7.55	12.60	22.75	38.25
CM		18.75	28.85	41.45	54.50
LR	20,10,20,10,20,10	9.05	14.50	24.30	36.10
ML	10,5,10,5	12.40	19.70	31.25	44.20
MB		12.40	19.70	31.25	44.20
EP		4.65	6.50	11.35	17.15
CM		13.00	17.60	23.65	31.05
LR	20,10,20,10,20,10	17.00	24.35	37.05	49.55
ML	10,10,10,10	17.00	24.35	37.05	49.55
MB		17.00	24.35	37.05	49.55
EP		14.75	23.85	37.85	51.65
CM		15.70	19.70	25.85	32.35
LR	30,10,25,10,20,10	15.70	23.40	34.10	46.30
ML	15,10,10,10	15.70	23.40	34.10	46.30
MB		15.70	23.40	34.10	46.30
EP		14.60	23.55	36.70	50.45
CM		15.80	20.25	25.40	30.60
LR	30,20,25,20,20,15	12.05	28.05	50.60	69.85
ML	15,10,10,5	19.25	37.95	61.20	79.35
MB		19.25	37.95	61.20	79.35
EP		3.55	6.35	12.15	21.35
CM		20.20	34.45	52.10	69.55

Table 4.8: Empirical powe. (%) of the statistics LR, ML, MB, EP and CM corresponding to nominal size $\alpha = 0.10$; critical values based on 10,000 replications; power based on 2000 replications; L = 5.

Tests	n_1,r_1,n_2,r_2,n_3,r_3 n_4,r_4,n_5,r_5	$(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ (1,1.2,1.4,1.6,1.8)	(1,1.6,1.8,2.2.2)	(1,2.2.2.2.4,2.6)	(1,2.4,2.6,2.8,3)
LR	5,5,5,5,5,5	15.05	19.15	24.70	30.10
ML	5,5,5,5	15.05	19.15	24.70	30.10
MB		15.05	19.15	24.70	30.10
EP		14.45	18.60	24.55	30.95
CM		15.25	17.30	20.25	23.20
LR	10,10,10,10,10,10	23.75	33.95	47.45	60.45
ML	10,10,10,10	23.75	33.95	47.45	60.45
MB		23.75	33.95	47.45	60.45
EP		22.85	34.65	48.65	62.60
CM		23.90	31.15	38.95	47.05
LR	10,7,10,7,10,7	19.55	25.50	33.20	43.25
ML	10,7,10,7	19.55	25.50	33.20	43.30
MB		19.55	25.50	33.20	43.30
EP		19.10	25.75	34.15	44.40
CM		18.70	23.45	27.85	33.10
LR	10,5,10,5,10,5	15.90	19.70	24.60	30.75
ML	10,5,10,5	15.90	19.70	24.60	30.75
MB		15.90	19.70	24.60	30.75
EP		15.90	19.80	25.55	32.50
CM		15.15	17.85	20.30	23.05
LR	20,20,20,20,20,20	45.30	67.25	84.75	93.55
ML	20,20,20,20	45.30	67.35	84.75	93.55
MB		45.30	67.35	84.75	93.55
EP		42.95	66.70	85.75	94.30
CM		44.70	61.85	78.20	88.85
LR	20,15,20,15,20,15	35.15	53.55	71.55	83.70
ML	20,15,20,15	35.15	53.55	71.55	83.70
MB		35.15	53.55	71.55	83.70
EP		32.65	54.05	73.60	86.10
CM		35.00	48.30	62.10	74.45
LR	20,10,20,10,20,10	26.85	38.05	50.70	64.05
ML	20,10,20,10	26.85	38.05	50.70	64.05
MB		26.85	38.05	50.70	64.05
EP		25.00	36.80	51.20	67.35
CM		26.85	33.75	42.20	51.00
LR	20,5,20,5,20,5	16.35	20.70	25.75	31.85
ML	20,5,20,5	16.35	20.70	25.75	31.85
MB		16.35	20.70	25.75	31.85
EP		14.70	18.75	25.20	31.80
CM		16.40	19.25	21.55	24.60

Table 4.8 continued

LR	10,10,10,10,10,10	36.00	46.90	58.80	71.25
ML	20,20,20,20	31.50	41.95	54.25	67.55
MB		31.50	41.95	54.25	67.55
EP		31.60	45.50	61.95	75.55
CM		30.20	36.95	44.95	52.55
LR	10,7,10,7,10,7	29.10	36.95	45.90	55.15
ML	20,15,20,15	23.65	30.85	40.00	50.15
MB		23.65	30.85	40.00	50.15
EP		24.90	34.65	47.00	58.75
CM		22.25	26.40	30.90	35.30
LR	10,5,10,5,10,5	23.15	27.80	33.50	40.25
ML	20,10,20,10	19.35	23.75	28.65	35.50
MB		19.35	23.75	28.65	35.50
EP		20.10	25.90	34.30	42.15
CM		18.40	19.85	22.45	24.80
LR	20,20,20,20,20,20	30.70	54.65	76.45	89.40
ML	10,10,10,10	35.70	59.60	79.95	91.50
MB		35.70	59.60	79.95	91.50
EP		22.55	43.35	65.40	81.80
CM		35.60	54.75	73.35	85.50
LR	20,15,20,15,20,15	22.90	38.85	58.70	73.60
ML	10,7,10,7	28.00	46.00	64.65	78.10
MB		28.00	46.00	64.65	78.10
EP		16.10	25.45	42.00	58.55
CM		28.30	41.70	57.30	69.50
LR	20,10,20,10,20,10	15.85	24.55	37.45	50.55
ML	10,5,10,5	21.20	30.85	45.25	58.75
MB		21.20	30.85	45.25	58.80
EP		10.75	16.10	23.25	33.25
CM		21.10	28.55	37.85	48.20
LR	20,10,20,10,20,10	25.70	36.80	51.35	64.60
ML	10,10,10,10	25.70	36.80	51.35	64.60
MB		25.70	36.80	51.35	64.60
EP		24.60	37.50	52.40	67.40
CM		25.00	33.05	41.80	51.35
LR	30,10,25,10,20,10	25.25	35.90	48.75	62.95
ML	15,10,10,10	25.25	35.90	48.75	62.95
MB		25.25	35.90	48.75	62.90
EP		24.05	35.90	50.60	65.50
CM		24.55	30.35	39.30	48.60
LR	30,20,25,20,20,15	20.10	39.80	63.60	81.35
ML	15,10,10,5	28.40	50.00	72.70	87.25
MB		28.40	50.00	72.70	87.25
EP		9.80	16.65	29.30	44.00
CM		30.35	46.80	67.35	82.55

CHAPTER 5

TESTING HYPOTHESES FOR MULTIPLE SAMPLES FROM TWO PARAMETER EXTREME VALUE DISTRIBUTIONS

5.1 INTRODUCTION

In the previous two chapters, we developed and studied procedures for testing the equality of several scale parameters based on samples from the two parameter gamma and exponential distributions. In this chapter we deal with testing the equality of several scale parameters in presence of a common shape parameter for failure censored samples from Weibull distributions having pdf (2.13.4). Equivalently and more conveniently we deal with testing the equality of several location parameters in the presence of common scale parameter for type II censored samples from extreme value distributions with pdf as in (2.13.5)

$$f(X;\ u,b) = \frac{1}{b}\exp\left\{\left(\frac{X-u}{b}\right) - \exp\left(\frac{X-u}{b}\right)\right\}, \tag{5.1.1}$$

where u and b are the location and scale parameters respectively. The Weibull distribution is one of the most widely used distributions, particularly in the fields of engineering, manufacturing, aeronautics and bio-medical sciences. Various problems associated with both the Weibull and the extreme value distributions have been considered by many authors, among whom are Cohen (1965), Harter and Moore (1965, 1967), Kimball (1946), Engelhardt and Bain (1973, 1974, 1977, 1981) and McCool (1979, 1982). Many of the available results about these distributions are reviewed by Lawless (1982), Nelson (1982) and Mann, Schefer and Singpurwalla (1974). As we described in chapter 2, since u and b are location- scale parameters, it is easier to work with the extreme value distribution given by (5.1.1).

In the statistical literature, many estimators have been proposed for the extreme value location- scale parameters based on failure censored data. For example, best linear unbiased estimators for these parameters are developed by Lieblein and Zelen (1956); best linear invariant estimators are considered by Mann (1967) and Mann and Fertig (1973). For the computation of these estimators, the necessary coefficients are tabulated by Mann (1967) for samples of size up to 25. Another set of estimators called the simple linear estimators are proposed by Bain (1972) and modified by Engelhardt and Bain (1973, 1974). One of the most reliable estimators is the maximum likelihood estimator (MLE) because of its desirable properties such as consistency, asymptotic normality and asymptotic efficiency. It is applicable to most statistical models and to most types of data. The maximum likelihood estimators of the parameters of the Weibull or extreme value distributions are not easily obtained since they usually involve the numerical solutions of a system of non- linear equations. However, routines for solving such non- linear equations are readily available in subroutine libraries such as IMSL and NAG.

Often lifetime data are collected in the form of multiple samples assumed to have come from extreme value distribution with common scale parameter b (or equivalently from Weibull distribution with common shape parameter β) (Nelson, 1970). This situation is analogous to assuming common variance in normal theory analysis of variance. If the scale parameters can be assumed equal then the test of equality of location parameters (or equivalently testing homogeneity of the Weibull scale parameters) is equivalent to testing the equality of reliabilities at a certain time. This can be seen from the definition of reliability function given in section 2.13.3. Lawless (1982) proposed a likelihood ratio test statistic for testing the equality of several location parameters in the presence of a common scale parameter b. McCool (1979, 1982) derived a statistic, namely, the shape parameter ratio statistic, which is based on the ratio of the two maximum likelihood estimators of the scale parameters with and without assuming different location parameters across the populations, and compared with the likelihood ratio statistic. Nelson (1982) proposed a general purpose quadratic test of homogeneity. In various situations we found that this statistic is very anti- conservative. So, we do not consider it here or elsewhere. For testing the above hypothesis, we derive a $C(\alpha)$ statistic and conduct extensive Monte Carlo studies to examine the behaviour of this statistic and the statistics proposed by Lawless (1982) and McCool (1979, 1980), in terms of size and power.

The above three test procedures have been developed with the assumption of common scale parameter b. It may be of concern to test whether the assumption of a common b is valid. For this purpose, Lawless and Mann (1976) proposed a modified likelihood ratio statistic (Bartlett's statistic) and a marginal likelihood ratio statistic. McCool (1979) derived an extremal scale parameter ratio statistic whose null distribution depends only on the sample size (n), the number of failures (r) and the number of groups (L). In this case, we derive a $C(\alpha)$ statistic and then compare the performance of all these statistics in terms of size and power by conducting a simulation study.

In section 5.2, we describe and develop estimators of the parameters under different null and alternative hypotheses to be tested in sections 5.3 and 5.6. Section 5.3 presents the procedures for the tests of homogeneity of location parameters in the presence of common scale parameter b, and section 5.4 presents the simulation study and results. An adjustment to the $C(\alpha)$ statistic, to hold the nominal level, is developed and validated in section 5.5. Section 5.6 studies the procedures for testing the assumption of common scale parameter and section 5.7 reports the simulation study and results. Two examples are given in section 5.8. Expected mixed partial derivatives for the derivation of the $C(\alpha)$ statistics are given in section 5.9.

5.2 ESTIMATION

5.2.1 Maximum Likelihood Estimation

Consider L samples from Weibull distributions with parameters $(\alpha_1,\beta_1),\ldots,(\alpha_L,\beta_L)$ or equivalently L samples from extreme value distributions with parameters $(u_1,b_1),\ldots,(u_L,b_L)$. Let t_{ij} or X_{ij} ($= \log t_{ij}$) denote the jth ordered observation in a sample of size n_i drawn from the ith population. It is assumed that n_i items from the ith population are tested until the r_ith item has failed ($r_i \le n_i$, $i = 1,\ldots,L$). Thus, if $r_i = n_i$, $i = 1,\ldots,L$, we deal with complete samples. For testing the hypothesis of equality of the extreme value location parameters with common scale parameter b, the competing hypotheses are

H_0: $u_1 = \ldots = u_L$ ($= u$)

and

H_1: not all u_i's are equal for all $b > 0$.

This is equivalent to testing the equality of Weibull scale parameters in the presence of a common shape parameter β; that is

$$H_0^*: \quad \alpha_1 = ... = \alpha_L \ (= \alpha)$$

and

$$H_1^*: \quad \text{not all } \alpha_i\text{'s are equal for all } \beta > 0.$$

For testing the hypothesis of equality of the extreme value scale parameters in the presence of unspecified location parameters, the competing hypotheses are

$$H_1: \quad b_1 = ... = b_L \ (= b), \ u_1,...,u_L \text{ are unspecified and possibly not equal}$$

and

$$H_2: \quad \text{not all } b_i\text{'s are equal, } u_1,...,u_L \text{ are unspecified and possibly not equal.}$$

This is equivalent to testing the equality of Weibull shape parameters in the presence of unspecified scale parameters $\alpha_1,...,\alpha_L$; that is

$$H_1^*: \quad \beta_1 = ... = \beta_L \ (= \beta)$$

and

$$H_2^*: \quad \text{not all } \beta_i\text{'s are equal, } \alpha_1,...,\alpha_L \text{ are unspecified.}$$

For convenience, derivations and results presented in this chapter are based on extreme value distributions.

Denote $\mathbf{u} = (u_1,...,u_L)'$ and $\mathbf{b} = (b_1,...,b_L)'$. From section 2.12.1, the log likelihood function $L(\mathbf{u},\mathbf{b})$ of a random sample $\{ X_{ij} \}$, $j = 1,...,r_i$, from the distribution with pdf (5.1.1) is, apart from a constant term, given by

$$l(u,b) = \sum_{i=1}^{L} \left\{ -r_i \log b_i + \sum_{j=1}^{r_i} \left(\frac{X_{ij}-u_i}{b_i} \right) - \sum_{j=1}^{*} \exp\left(\frac{X_{ij}-u_i}{b_i} \right) \right\}, \tag{5.2.1}$$

where we use the notation $\sum_{j=1}^{*} W_{ij} = \sum_{j=1}^{r} W_{ij} + (n_i - r_i) W_{ir_i}$, for a sequence

$W_{i1}, \ldots, W_{ir_i}$, $i = 1,..,L$, as stated in chapter 4.

Under the hypothesis H_2, for $i = 1,\ldots,L$, solution of the likelihood equations $\partial l/\partial b_i = 0$ and $\partial l/\partial u_i = 0$ yields the maximum likelihood estimates $\hat{b}_i$ and $\hat{u}_i$ of b_i and u_i respectively.

Now,

$$\frac{\partial l}{\partial u_i} = -\frac{1}{b_i} \left\{ r_i - \sum_{j=1}^{*} \exp\left(\frac{X_{ij}-u_i}{b_i} \right) \right\},$$

and

$$\frac{\partial l}{\partial b_i} = -\frac{1}{b_i} \left\{ r_i + \sum_{j=1}^{r_i} \left(\frac{X_{ij}-u_i}{b_i} \right) - \sum_{j=1}^{*} \left(\frac{X_{ij}-u_i}{b_i} \right) \exp\left(\frac{X_{ij}-u_i}{b_i} \right) \right\}.$$

It is easily seen that $\partial l/\partial u_i = 0$ implies

$$\exp(\hat{u}_i) = \left[\frac{1}{r_i} \sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{\hat{b}_i} \right) \right]^{\hat{b}_i}.$$

Substituting the expression for $\hat{u}_i$ into $\partial l/\partial b_i = 0$, we obtain

$$\frac{\sum_{j=1}^{*} X_{ij} \exp\left(\frac{X_{ij}}{\hat{b}_i}\right)}{\sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{\hat{b}_i}\right)} - \bar{X}_{i.} - \hat{b}_i = 0, \tag{5.2.2}$$

where $\bar{X}_{i.} = \sum X_{ij}/r_i$, $j = 1,\ldots,r_i$; $i = 1,\ldots,L$. The equation (5.2.2) involves only $\hat{b}_i$ and thus it can be solved for $\hat{b}_i$. Once $\hat{b}_i$ is obtained, the MLE $\hat{u}_i$ of u_i, again under H_2, is

$$\hat{u}_i = \hat{b}_i \log\left[\frac{1}{r_i}\sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{\hat{b}_i}\right)\right]. \tag{5.2.3}$$

Under the hypothesis H_1, the log likelihood function (5.2.1) reduces to

$$l(u,b) = \sum_{i=1}^{L}\left\{-r_i \log b + \sum_{j=1}^{r_i}\left(\frac{X_{ij}-u_i}{b}\right) - \sum_{j=1}^{*} \exp\left(\frac{X_{ij}-u_i}{b}\right)\right\}. \tag{5.2.4}$$

Differentiating and equating partial derivatives $\partial l/\partial u_i$ and $\partial l/\partial b$ to zero, we obtain

$$\sum_{j=1}^{*} \exp\left(\frac{X_{ij}-u_i}{b}\right) - r_i = 0 \quad , \quad i = 1,\ldots,L$$

and

$$\sum_{i=1}^{L}\left\{\sum_{j=1}^{*}\left(\frac{X_{ij}-u_i}{b}\right)\exp\left(\frac{X_{ij}-u_i}{b}\right) - \sum_{j=1}^{r_i}\left(\frac{X_{ij}-u_i}{b}\right) - r_i\right\} = 0 .$$

As shown earlier, u_i, $i = 1,\ldots,L$, can be eliminated from the second equation $\partial l/\partial b = 0$ to give

$$\sum_{i=1}^{L} \left\{ \frac{r_i \sum_{j=1}^{*} X_{ij} \exp\left(X_{ij}/b\right)}{\sum_{j=1}^{*} \exp\left(X_{ij}/b\right)} - \sum_{j=1}^{r_i} X_{ij} - b\, r_i \right\} = 0 . \tag{5.2.5}$$

It is quite easy to solve the equation (5.2.5) iteratively for $\hat{b}$, the MLE of b under H_1. Subsequently, $\hat{u}_{ic}$, the MLE of u_i, follows from the equation $\partial l/\partial u_i = 0$, i = 1,...,L, as

$$\hat{u}_{ic} = \hat{b} \log \left[\frac{1}{r_i} \sum_{j=1}^{*} \exp\left(X_{ij}/\hat{b}\right) \right]. \tag{5.2.6}$$

Under the hypothesis H_0, the log likelihood function (5.2.1) becomes

$$l(u,b) = \sum_{i=1}^{L} \left\{ - r_i \log b + \sum_{j=1}^{r_i} \left(\frac{X_{ij}-u}{b} \right) - \sum_{j=1}^{*} \exp\left(\frac{X_{ij}-u}{b} \right) \right\}. \tag{5.2.7}$$

Differentiating the log likelihood function (5.2.7) with respect to u and b, and equating partial derivatives $\partial l/\partial u$ and $\partial l/\partial b$ to zero, we obtain the ML equations

$$\sum_{i=1}^{L} \left\{ \sum_{j=1}^{*} \exp\left(\frac{X_{ij}-u}{b} \right) - r_i \right\} = 0 ,$$

and

$$\sum_{i=1}^{L} \left\{ \sum_{j=1}^{*} \left(\frac{X_{ij}-u}{b} \right) \exp\left(\frac{X_{ij}-u}{b} \right) - \sum_{j=1}^{r_i} \left(\frac{X_{ij}-u}{b} \right) - r_i \right\} = 0 .$$

On eliminating u from the second equation, we obtain the following equation, which can be solved for $\hat{b}_c$, the MLE of b, iteratively.

$$\frac{\sum_{i=1}^{L} \sum_{j=1}^{*} X_{ij} \exp\left(X_{ij}/\hat{b}_c\right)}{\sum_{i=1}^{L} \sum_{j=1}^{*} \exp\left(X_{ij}/\hat{b}_c\right)} - \bar{X}_{..} - \hat{b}_c = 0, \tag{5.2.8}$$

where

$$\bar{X}_{..} = \frac{1}{R} \sum_{i=1}^{L} \sum_{j=1}^{r_i} X_{ij} \, ; \quad R = \sum_{i=1}^{L} r_i \, .$$

From the equation $\partial l/\partial u = 0$, $\hat{u}_c$, the MLE of u follows as

$$\hat{u}_c = \hat{b}_c \log \left\{ \frac{1}{R} \sum_{i=1}^{L} \sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{\hat{b}_c} \right) \right\} . \tag{5.2.9}$$

5.2.2 Maximum Marginal Likelihood Estimation

Following the theory in section 2.8.2, the log marginal likelihood function $l_m(\mathbf{b})$ for the scale parameter **b**, under H_2, is given by

$$l_m(b) = \sum_{i=1}^{L} \left\{ -(r_i-1)\log b_i + \frac{X_{i.}}{b_i} - r_i \log\left(\sum_{j=1}^{*} \exp(X_{ij}/b_i) \right) \right\}. \tag{5.2.10}$$

Differentiating the log marginal likelihood $l_m(\mathbf{b})$ with respect to the parameters b_i and equating the partial derivatives $\partial l/\partial b_i$ to zero we obtain , for i = 1,...,L,

$$\frac{r_i \sum_{j=1}^{*} X_{ij} \exp\left(\frac{X_{ij}}{b_i}\right)}{\sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{b_i}\right)} - X_{i.} - (r_i-1)b_i = 0. \tag{5.2.11}$$

Solutions of these equations yield the maximum marginal likelihood estimates (MMLE)

$\tilde{b}_i$ for b_i, $i = 1,\ldots,L$, under the hypothesis H_2.

Under the hypothesis H_1, the log marginal likelihood function reduces to

$$l_m(b) = -\sum_{i=1}^{L}\left\{(r_i-1)\log b - \left(\frac{X_{i.}}{b}\right) + r_i \log\left(\sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{b}\right)\right)\right\}. \tag{5.2.12}$$

The consequent maximum likelihood equation for b is

$$\sum_{i=1}^{L}\left\{\frac{r_i \sum_{j=1}^{*} X_{ij}\exp\left(X_{ij}/b\right)}{\sum_{j=1}^{*}\exp\left(X_{ij}/b\right)} - X_{i.} - (r_i-1)b\right\} = 0\ , \tag{5.2.13}$$

which can be solved iteratively for $\tilde{b}$, the maximum marginal likelihood estimate for b, under the hypothesis H_1.

5.3 TESTING EQUALITY OF SEVERAL LOCATION PARAMETERS IN PRESENCE OF A COMMON SCALE PARAMETER b

5.3.1 Likelihood Ratio Statistic (LRu)

The hypothesis of interest is H_0: $u_1 = \ldots = u_L$ and the alternative hypothesis H_1: not all u_i's are equal. Using the maximum likelihood estimators of the parameters, under H_0 and H_1, given in section 5.2, the maximized log likelihood function l_1, under H_1, is,

$$l_1 = \sum_{i=1}^{L}\left\{\sum_{j=1}^{r_i}\left(\frac{X_{ij}-\hat{u}_{ic}}{\hat{b}}\right) - r_i \log \hat{b}\right\}$$

and, under that H_0 is

$$l_0 = \sum_{i=1}^{L} \left\{ \sum_{j=1}^{r_i} \left(\frac{X_{ij}-\hat{u}_c}{\hat{b}_c} \right) - r_i \log \hat{b}_c \right\}.$$

Hence the log likelihood ratio statistic (LRu) is given by

$$LRu = 2 \left\{ \sum_{i=1}^{L} \sum_{j=1}^{r_i} \left[\left(\frac{X_{ij}-\hat{u}_{ic}}{\hat{b}} \right) - \left(\frac{X_{ij}-\hat{u}_c}{\hat{b}_c} \right) \right] + R \log\left(\hat{b}_c/\hat{b} \right) \right\}, \tag{5.3.1}$$

where $R = \sum r_i$, $i = 1,\dots,L$. Under the null hypothesis H_0, the distribution of the statistic LRu is approximately distributed as chi-square with (L-1) degrees of freedom.

5.3.2 Shape Parameter Ratio Statistic (SP)

McCool (1979, 1982) proposed a statistic based on the ratio of the estimates of the Weibull shape parameters under the alternative and the null hypotheses for testing the equality of Weibull scale parameters in the presence of a common shape parameter. He notes that under the null hypothesis of equal scale parameters the distribution of the ratio of the estimators of the shape parameters is parameter free and hence it can be considered a pivotal quantity. In terms of extreme value distribution scale parameters, the statistic denoted by SP is

$$SP = \hat{b}_c/\hat{b}. \tag{5.3.2}$$

Under the null hypothesis, the distribution of the statistic SP is parameter free and therefore can be used as a test statistic for testing H_0. However, its distribution is unknown and the critical values need to be evaluated empirically.

5.3.3 C(α) Statistic (CLu)

As in section 3.3.3, suppose that the alternative hypothesis H_1 is written as $u_i =$

$u + \phi_i$ with $\phi_L = 0$. Then testing hypothesis H_0 is equivalent to testing H_0: $\phi_i = 0$ for all i w th u and b treated as nuisance parameters. This reparametrization is more convenient for the derivation of the C(α) test.

Now, the log likelihood (5.2.4) in terms of ϕ_i, i = 1,...,L, u and b is

$$l = \sum_{i=1}^{L} \left\{ - r_i \log b + \sum_{j=1}^{r_i} \left(\frac{X_{ij} - u - \phi_i}{b} \right) - \sum_{j=1}^{*} \exp\left(\frac{X_{ij} - u - \phi_i}{b} \right) \right\}, \tag{5.3.3}$$

Denote $\phi = (\phi_1, ..., \phi_{L-1})'$ and $\theta = (\theta_1, \theta_2)' = (u, b)'$.

5.3.3.1 C(α) Statistic for Complete Samples

Based on the likelihood l in (5.3.3), we obtain,

for i = 1,...,L,

$$\psi_i = \frac{\partial l}{\partial \phi_i}\bigg|_{\phi=0} = \frac{1}{b} \left\{ \sum_{j=1}^{n_i} \exp\left(\frac{X_{ij} - u}{b} \right) - n_i \right\},$$

$$\eta_1 = \frac{\partial l}{\partial \theta_1}\bigg|_{\phi=0} = \frac{\partial l}{\partial u}\bigg|_{\phi=0} = \frac{1}{b} \left\{ \sum_{i=1}^{L} \sum_{j=1}^{n_i} \exp\left(\frac{X_{ij} - u}{b} \right) - N \right\}$$

and

$$\eta_2 = \frac{\partial l}{\partial \theta_2}\bigg|_{\phi=0} = \frac{\partial l}{\partial b}\bigg|_{\phi=0} = \frac{1}{b} \left\{ \sum_{i=1}^{L} \sum_{j=1}^{n_i} \left(\frac{X_{ij} - u}{b} \right) \left[\exp\left(\frac{X_{ij} - u}{b} \right) - 1 \right] - N \right\}.$$

Further, from section 5.9, we have

$$D_{ij} = -E\left(\frac{\partial^2 l}{\partial\phi_i \partial\phi_j} \Bigg|_{\phi=0} \right) = \begin{cases} \dfrac{n_i}{b^2} & , \quad 1 \le i = j \le (L-1) \\ 0 & , \quad 1 \le i \ne j \le (L-1) \end{cases} ,$$

$$A_{ij} = -E\left(\frac{\partial^2 l}{\partial\phi_i \partial\theta_j} \Bigg|_{\phi=0} \right) = \begin{cases} \dfrac{n_i}{b^2} & , \quad j=1;\ i=1,\ldots,(L-1) \\ \dfrac{n_i\,(1-\gamma)}{b^2} & , \quad j=2;\ i=1,\ldots,(L-1) \end{cases} ,$$

and

$$B_{jj'} = -E\left(\frac{\partial^2 l}{\partial\theta_j \partial\theta_{j'}} \Bigg|_{\phi=0} \right) = \begin{cases} \dfrac{N}{b^2} & , \quad j = j' = 1 \\ \dfrac{N\,K}{b^2} & , \quad j = j' = 2 \\ \dfrac{N\,(1-\gamma)}{b^2} & , \quad j \ne j'\ ; j, j' = 1,2, \end{cases}$$

where $K = \pi^2/6 + (1-\gamma)^2$ and γ is the Euler's constant.

Now, the nuisance parameters u and b in ψ_i, η_1, η_2, D_{ij}, A_{ij} and $B_{jj'}$ are replaced by their MLEs. Then, following the general theory in section 2.15.2, the $C(\alpha)$ statistic is

$CLu = \hat{\psi}'\, V^{-1}\, \hat{\psi}$, where $V = D - A\, B^{-1}\, A'$.

After some simplification, we obtain the (i,j)th element of V as

$$V_{ij} = \begin{cases} \dfrac{N\, p_i\,(1-p_i)}{b^2} \;, & 1 \le i = j \le (L-1) \\[2ex] \dfrac{-\,N\, p_i\, p_j}{b^2} \;, & 1 \le i \ne j \le (L-1) \;, \end{cases}$$

where $p_i = n_i/N$, $i = 1,\ldots,L$. After further simplification the $C(\alpha)$ statistic is obtained as

$$CLu = \sum_{i=1}^{L} \frac{1}{n_i} \left\{ \sum_{j=1}^{n_i} \exp\left(\frac{X_{ij}-\hat{u}}{\hat{b}} \right) - 1 \right\}^2 , \tag{5.3.4}$$

which is asymptotically distributed as chi-square with (L-1) degrees of freedom.

5.3.3.2. C(α) Statistic for Censored Samples

For type II censored samples, we have

$$\psi_i = \left. \frac{\partial l}{\partial \phi_i} \right|_{\phi=0} = \frac{1}{b} \left\{ {\sum_{j=1}}^{*} \exp\left(\frac{X_{ij}-u}{b} \right) - r_i \right\} ,$$

$$\eta_1 = \left. \frac{\partial l}{\partial \theta_1} \right|_{\phi=0} = \left. \frac{\partial l}{\partial u} \right|_{\phi=0} = \frac{1}{b} \left\{ \sum_{i=1}^{L} {\sum_{j=1}}^{*} \exp\left(\frac{X_{ij}-u}{b} \right) - R \right\}$$

and

$$\eta_2 = \frac{\partial l}{\partial \theta_2}\bigg|_{\phi=0} = \frac{\partial l}{\partial b}\bigg|_{\phi=0}$$

$$= \frac{1}{b}\sum_{i=1}^{L}\left\{\sum_{j=1}^{*}\left(\frac{X_{ij}-u}{b}\right)\exp\left(\frac{X_{ij}-u}{b}\right) - \sum_{j=1}^{r_i}\left(\frac{X_{ij}-u}{b}\right) - r_i\right\}.$$

From section 5.9, we have for i = 1,...,(L-1),

$$D_{ij} = -E\left(\frac{\partial^2 l}{\partial\phi_i\partial\phi_j}\bigg|_{\phi=0}\right) = \begin{cases} \dfrac{r_i}{b^2} & , \quad 1 \le i = j \le (L-1) \\ 0 & , \quad 1 \le i \ne j \le (L-1) \end{cases},$$

$$A_{ij} = -E\left(\frac{\partial^2 l}{\partial\phi_i\partial\theta_j}\bigg|_{\phi=0}\right) = \begin{cases} \dfrac{r_i}{b^2} & , \quad j=1; i=1,\dots,(L-1) \\ \dfrac{I_i}{b^2} & , \quad j=2; i=1,\dots,(L-1) \end{cases},$$

and

$$B_{jj'} = -E\left(\frac{\partial^2 l}{\partial\theta_j\partial\theta_{j'}}\bigg|_{\phi=0}\right) = \begin{cases} \dfrac{R}{b^2} & , \quad j=j'=1 \\ \displaystyle\sum_{i=1}^{L}\frac{J_i}{b^2} & , \quad j=j'=2 \\ \displaystyle\sum_{i=1}^{L}\frac{I_i}{b^2} & , \quad j\ne j';\ j,j'=1,2. \end{cases},$$

where

$$I_i = \sum_{j=1}^{*} \left(t_j \log t_j + \frac{d_j}{2t_j} \right)$$

and

$$J_i = \sum_{j=1}^{*} (2 + \log t_j)\left(t_j \log t_j + \frac{d_j}{t_j} \right) - \sum_{j=1}^{r_i} \left(2 \log t_j - \frac{d_j}{t_j^2} \right) - r_i \, .$$

Now, the nuisance parameters u and b in ψ_i, η_1, η_2, D_{ij}, A_{ij} and B_{ij}, are replaced by their MLEs. Then, following section 2.16.2, the $C(\alpha)$ statistic for censored samples is

$$CLu = \hat{\psi}' (D - AB^{-1}A') \, \hat{\psi}. \tag{5.3.5}$$

5.4 SIMULATION STUDY

A simulation study was conducted to compare the empirical size of the statistics LRu and CLu whose asymptotic distribution is known as $\chi^2(L\text{-}1)$. The shape parameter ratio statistic SP, which will be included in the power study, was not included here as its asymptotic null distribution is not known. For each L, samples from extreme value distribution were generated through IMSL (1982) subroutine GGWIB. Without loss of generality we chose u = 0.0 and b = 0.3. Simulations were performed for L = 2, 3 and 5 using nominal levels α = 0.10, 0.05 and 0.01. Each experiment was based on 2000 replications. The results are reported in Table 5.1 for various combinations of (n,r), which represent the degree of censoring in each sample. To examine the power performance of the statistics LRu, CLu and SP, we conducted a simulation study again for L = 2, 3, 5; α = 0.10, 0.05, 0.01 and all combinations of (n,r) presented in Table 5.1. The results are reported in Tables 5.2 through 5.10. From Table 5.1, we can see that the null distribution of the statistics LRu and CLu vary widely unless the sample sizes are large. Also the distribution of the statistic SP is not known. Therefore, to examine the power performance of the statistics, we calculated critical values from the empirical distribution of all the statistics, based on 10,000 replications. These critical values were then used in the power study which was based on 2000 replications.

Results

Table 5.1 shows that the statistic LRu is too liberal for all n,r,L and α combinations that we investigated. The performance of this statistic worsens as the number of groups (L) and the percentage of censoring increases. The statistic CLu holds nominal level reasonably well for $\alpha = 0.10$, but it shows conservative behaviour with decreasing α and also with increasing percentage of censoring. Note that 2 times the standard error of the probabilities based on $\alpha = 0.01$, 0.05 and 0.10 are respectively, 0.005, 0.010 and 0.013. Empirical levels less than α - 2 standard error are termed as conservative and those greater than α + 2 standard error are termed as liberal.

From Tables 5.2 through 5.4, we can see that for fixed absolute values of (u_2-u_1), power of all three statistics increases with decreasing b. However, if the absolute relative value of $(u_2-u_1)/b$ remains fixed then the power of the statistics remains unchanged as b increases. Further, Tables 5.5 through 5.10 show that the power of all three statistics decreases as percentage of censoring increases. In situations where $n_1 = ... = n_L$ and $r_1 = ... = r_L$, L = 2, 3, 5, all three statistics have similar power. The statistic CLu is, in general, most powerful, particularly in situations where $n_1 > ... > n_L$ and $b_1 < ... < b_L$.

For situations $n_1 < ... < n_L$ and $b_1 < ... < b_L$ the power of the statistic CLu is found to be less than that of the other statistics. In both situations, some of the n_i's ($i = 1,...,L$) may be equal.

Overall, the statistic CLu is conservative for small sample size and for censored sample situations. However, when all the tests maintain nominal level, CLu is never less powerful than the others. So based on empirically computed critical values the statistic CLu is preferable. In the next section we propose an adjusted $C(\alpha)$ statistic (ACLu), which holds nominal level well and therefore will have same power properties as the CLu based on empirical critical values.

5.5 IMPROVEMENT OF THE STATISTIC CLu TO MAINTAIN LEVEL

The asymptotic distribution of CLu is $\chi^2(L-1)$, which we observed to be conservative for small samples. A common way to improve the performance is to adopt a $c\chi^2(d)$ distribution for CLu. Such a distribution for a $C(\alpha)$ statistic showed improved approximation in other situations; for example Dean and Lawless (1989). The constants c and d are obtained by equating the first two moments of $c\chi^2(d)$ and CLu. Let M and V be the mean and variance of CLu. Then $c = V/2M$ and $d = 2\,M^2/V$. Thus, the adjusted CLu, denoted by ACLu, is given by $ACLu = CLu/c \sim \chi^2(d)$. It is difficult to obtain analytic expressions for M and V. So, we conducted extensive simulations to find M and V. For each L (L is the number of groups), let $\dot{N}$ be the average sample size and $\dot{R}$ be the average number of lifetimes. For various combinations of $(L, \dot{N}, \dot{R})$, M and V were computed from the empirical distributions of CLu based on 2000 replications. We noticed that M varies only over L and $\dot{N}$, but V varies over L, $\dot{N}$ and $\dot{R}$. For each L (L = 2,...,10), we obtained M and V empirically for 116 points ($\dot{N}$, $\dot{R}$) = (5,3),...,(5,5), (7,3),...,(7,7),(10,3),...,(10,10),(12,3),...,(12,12),(15,3),...,(15,15),18,3),...,(18,18),(20,3),... ,(20,20),(22,3),...,(22,22),(25,3),...,(25,25). We believe the range of values of L, $\dot{N}$ and $\dot{R}$ taken here is sufficient for practical applications.

By fitting polynomial regression model for M in terms of L, $\dot{N}$ and $\dot{R}$, we found a single equation which depends only on L and $\dot{N}$, as

$$M = - 1.0133 + 0.9989\ L + 0.0022\ \dot{N}.$$

For V, no satisfactory single equation could be found which fit the 9x 116 = 1044 data points well. So, we fitted a regression equation for each L. For better fit we retained, in some cases, some terms even if they were insignificant. The equations for L = 2, ... , 10 are given in Table 5.11.

For fixed L, we take the average of the sample $\dot{N}$ and that of the lifetimes $\dot{R}$ and then compute M and V from the equations given in Table 5.11. Using these values of M and V, we calculate c and d. If ACLu is greater than $\chi^2(d)$ then we reject the null hypothesis H_0. Table 5.12 presents the empirical levels of the statistic ACLu for $\alpha = 0.10$, 0.05, 0.01; L = 2, 3, 5, 10; and various combinations of (n,r). Each experiment was based on 2000 replications.

At $\alpha = 0.01$, the statistic ACLu shows some conservative behaviour for small number of groups, small sample size and heavy censoring; otherwise it holds nominal level well. Note that at $\alpha = 0.10$, ACLu is slightly inflated for L = 2,3.

5.6 TESTING EQUALITY OF SCALE PARAMETERS IN PRESENCE OF UNSPECIFIED LOCATION PARAMETERS

In section 5.3 for testing equality of location parameters we assumed that the scale parameters are the same across all the extreme value populations. However, this assumption should be checked before testing the homogeneity of location parameters. For this problem, the competing hypotheses are

H_1: $b_1 = \ldots = b_L \ (= b\ )$

against

H_2: not all b_i's are equal, with unspecified $u_1,\ldots,u_L$.

5.6.1 Likelihood Ratio Statistic (LRb)

Using the maximum likelihood estimators of the parameters under the hypotheses H_1 and H_2 given in section 5.2, the maximized log likelihood function, under H_2, is given by

$$l_2 = \sum_{i=1}^{L} \left\{ \sum_{j=1}^{r_i} \left(\frac{X_{ij} - \hat{u}_i}{\hat{b}_i} \right) - r_i \log \hat{b}_i \right\}$$

and under H_1, the maximized log likelihood function is given by

$$l_1 = \sum_{i=1}^{L} \left\{ \sum_{j=1}^{r_i} \left(\frac{X_{ij} - \hat{u}_{ic}}{\hat{b}} \right) - r_i \log \hat{b} \right\}.$$

Thus, the log likelihood ratio statistic, as discussed in section 2.15.1, is given by

Under the null hypothesis H_1, the distribution of the statistic LRb is approximately chi-square with (L-1) degrees of freedom.

$$LRb = 2\sum_{i=1}^{L}\left\{\sum_{j=1}^{r_i}\left[\left(\frac{X_{ij}-\hat{u}_i}{\hat{b}_i}\right)-\left(\frac{X_{ij}-\hat{u}_{ic}}{\hat{b}}\right)\right]+r_i\log\left(\frac{\hat{b}}{\hat{b}_i}\right)\right\}. \tag{5.6.1}$$

5.6.2 Modified Likelihood Ratio Statistic (MB)

Lawless (1974) gave an approximation to the pdf of $\hat{b}/b$ in a complete single sample situation where $\hat{b}$ is the MLE of b. The approximation, which was developed empirically, is of the form $g(\hat{b}/b) \sim \chi^2(h)$ for type II censored sample. The constants $g = g(r,n)$; $h = h(r,n)$, where the first r lifetimes were observed from a sample of size n, are obtained by simulation studies, and are tabulated for some combinations of (n,r) by Lawless and Mann (1976) and Lawless (1982). Based on this approximation, Lawless and Mann (1976) developed a modified likelihood ratio statistic, using a Bartlett type correction, which is given by

$$MB = \frac{1}{K^*}\left[h\log b^* - \sum_{i=1}^{L} h_i \log\left(\frac{g_i\,\hat{b}_i}{h_i}\right)\right], \tag{5.6.2}$$

where

$$g_i = h_i + 2;\ K^* = 1 + \frac{1}{3(L-1)}\left(\sum_{i=1}^{L}\frac{1}{h_i} - \frac{1}{h}\right),$$

$$h_i = h(r_i,n_i);\ h = \sum_{i=1}^{L} h_i;\ b^* = \sum_{i=1}^{L}\frac{g_i\,\hat{b}_i}{h}.$$

The statistic MB is also approximately distributed as $\chi^2(L-1)$. The constants h_i for given values of r_i and n_i are given in Table 4.1.2 of Lawless (1982).

5.6.3 Marginal Likelihood Ratio Statistic (ML)

Lawless and Mann (1976) further studied a likelihood ratio statistic based on marginal likelihood to test the homogeneity of the extreme value scale parameters. Using the maximum marginal likelihood estimators of the scale parameters under H_1 and H_2 presented in section (5.2), the maximum log marginal likelihood function, under the hypothesis H_2, is given by

$$l_2^* = \sum_{i=1}^{L} \left\{ \frac{X_{i.}}{\bar{b}_i} - r_i \log\left(\sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{\bar{b}_i}\right) \right) - (r_i - 1) \log \bar{b}_i \right\}.$$

Under H_1, the maximum log marginal likelihood function is given by

$$l_1^* = \sum_{i=1}^{L} \left\{ \frac{X_{i.}}{\bar{b}} - r_i \log\left(\sum_{j=1}^{*} \exp\left(\frac{X_{ij}}{\bar{b}}\right) \right) - (r_i - 1) \log \bar{b} \right\}.$$

Thus, the log marginal likelihood ratio statistic can be written as

$$ML = 2\sum_{i=1}^{L} \left\{ (r_i - 1)\log(\bar{b}/\bar{b}_i) + \left(\frac{X_{i.}}{\bar{b}_i} - \frac{X_{i.}}{\bar{b}} \right) - r_i \log\left(\frac{\sum_{j=1}^{*} \exp(X_{ij}/\bar{b}_i)}{\sum_{j=1}^{*} \exp(x_{ij}/\bar{b})} \right) \right\}, \tag{5.6.3}$$

which is also approximately distributed as $\chi^2(L-1)$.

5.6.4 Extremal Scale Parameter Ratio Statistic (EP)

As discussed in previous chapters, the extremal scale parameter ratio statistic (EP), in terms of the estimates of the extreme value scale parameters, is given by

McCool (1979) showed that under the null hypothesis H_1, the statistic EP follows a distribution that depends only on n, r and L. He tabulated the results only for equal

$$EP = \frac{\underset{1 \le i \le L}{Max} \{\bar{b}_i\}}{\underset{1 \le i \le L}{Min} \{\bar{b}_i\}} \ . \tag{5.6.4}$$

sample size and equal number of failures in each group.

5.6.5 C(α) Statistic (CLb)

We follow the procedure as given in section 2.15.2. Suppose $b_i = b + \phi_i$ with $\phi_L = 0$, $i = 1,\dots,L$. Define $\phi = (\phi_1,\dots,\phi_{L-1})'$ and $\theta = (\theta_1,\dots,\theta_L,\theta_{L+1})' = (u_1,\dots,u_L,b)'$. Then for the derivations of the C(α) statistic (CLb) for complete samples the required quantities are given below.

$$\psi_i = \left.\frac{\partial l}{\partial \phi_i}\right|_{\phi=0} = \frac{G_i}{b} \ , \quad i=1,\dots,(L-1),$$

$$G_i = \sum_{j=1}^{n_i} \left\{ \left(\frac{X_{ij}-u_i}{b}\right)\left[\exp\left(\frac{X_{ij}-u_i}{b}\right) - 1 \right] - 1 \right\}, \qquad i=1,\dots,L,$$

$$\eta_k = \left.\frac{\partial l}{\partial \theta_k}\right|_{\phi=0} = \left.\frac{\partial l}{\partial u_k}\right|_{\phi=0} = \frac{H_k}{b} \ , \quad k=1,\dots,L,$$

$$H_k = \sum_{j=1}^{n_k} \left\{ \exp\left(\frac{X_{kj}-u_k}{b}\right) - 1 \right\},$$

and

$$\eta_{L+1} = \left.\frac{\partial l}{\partial \theta_{L+1}}\right|_{\phi=0} = \left.\frac{\partial l}{\partial b}\right|_{\phi=0} = \frac{G}{b}, \quad where \quad G = \sum_{i=1}^{L} G_i.$$

Further we define $K = \pi^2/6 + (1-\gamma)^2$. From section 5.9, we have

$$C_{ij} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial \phi_j}\right|_{\phi=0}\right) = \begin{cases} \dfrac{n_i K}{b^2} & , \quad i=j=1,\dots,(L-1), \\ 0 & , \quad i \neq j=1,\dots,(L-1), \end{cases} ,$$

$$E_{ik} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial \theta_k}\right|_{\phi=0}\right) = \begin{cases} \dfrac{n_i(1-\gamma)}{b^2} & , \quad i=k=1,\dots,(L-1), \\ 0 & , \quad i \neq k;\ k=1,\dots,L, \\ & , \quad k \neq i;\ i=1,\dots,(L-1), \\ \dfrac{n_i K}{b^2} & , \quad k=L+1;\ i=1,\dots,(L-1), \end{cases} ,$$

and

$$F_{kk'} = -E\left(\left.\frac{\partial^2 l}{\partial\theta_k \partial\theta_{k'}}\right|_{\phi=0}\right) = \begin{cases} \dfrac{n_k}{b^2} & , \quad k=k'=1,\ldots,L, \\ 0 & , \quad k\neq k'=1,\ldots,L, \\ \dfrac{n_k(1-\gamma)}{b^2} & , \quad k=L+1; k'=1,\ldots,L \\ & \quad\ \ k=1,\ldots,L; k'=L+1, \\ \dfrac{N\,K}{b^2} & , \quad k=k'=L+1. \end{cases}$$

When we use the MLEs of θ, in the expressions for ψ_i, η_k, C_{ij},E_{ik} and $F_{kk'}$, the C(α) statistic, after considerable steps of algebra, reduces to

$$CLb = \sum_{i=1}^{L} \frac{G_i^2}{n_i} = \sum_{i=1}^{L} \frac{1}{n_i}\left\{\sum_{j=1}^{n_i}\left(\frac{X_{ij}-\hat{u}_i}{\hat{b}}\right)\left[\exp\left(\frac{x_{ij}-\hat{u}_i}{\hat{b}}\right)-1\right]-1\right\}^2.$$

(5.6.5)

For the derivation of the C(α) statistic for censored samples the necessary quantities are given below.

$$\psi_i = \frac{1}{b}\left\{\sum_{j=1}^{*}\left(\frac{X_{ij}-u_i}{b}\right)\exp\left(\frac{X_{ij}-u_i}{b}\right) - \sum_{j=1}^{r_i}\left(\frac{X_{ij}-u_i}{b}\right) - r_i\right\},$$

$$C_{ij} = -E\left(\left.\frac{\partial^2 l}{\partial\phi_i \partial\phi_j}\right|_{\phi=0}\right) = \begin{cases} \dfrac{J_i}{b^2} & , \quad i=j=1,\ldots,(L-1), \\ 0 & , \quad i\neq j=1,\ldots,(L-1), \end{cases} ,$$

$$E_{ik} = -E\left(\left.\frac{\partial^2 l}{\partial\phi_i \partial\theta_k}\right|_{\phi=0}\right) = \begin{cases} \dfrac{I_i}{b^2} & , \quad i=k=1,\dots,(L-1), \\ 0 & , \quad i\neq k;\ k=1,\dots,L, \\ & , \quad k\neq i;\ i=1,\dots,(L-1), \\ \dfrac{J_i}{b^2} & , \quad k=L+1; i=1,\dots,(L-1), \end{cases}$$

and

$$F_{kk'} = -E\left(\left.\frac{\partial^2 l}{\partial\theta_k \partial\theta'_k}\right|_{\phi=0}\right) = \begin{cases} \dfrac{r_k}{b^2} & , \quad k=k'=1,\dots,L, \\ 0 & , \quad k\neq k';\ k'=1,\dots,L, \\ \dfrac{I_k}{b^2} & , \quad k=L+1;\ k'=1,\dots,L \\ & \quad\ \ k=1,\dots,L;\ k'=L+1, \\ \dfrac{\sum_{i=1}^{L} J_i}{b^2} & , \quad k=k'=L+1. \end{cases} ,$$

where the terms J_i's and I_i's are as defined in section 5.3.3.2. Now, we use the MLEs for θ, in the expressions of ψ_i, η_k, C_{ij},E_{ik} and $F_{kk'}$. Then the $C(\alpha)$ statistic reduces to

$$CLb = \hat{\psi}' \,(C - EF^{-1}E')^{-1}\, \hat{\psi}, \qquad (5.6.6)$$

which is asymptotically distributed as $\chi^2(L-1)$.

5.7 SIMULATION STUDY

A simulation study was conducted to investigate the behaviour of the statistics

LRb, MB, ML, EP and CLb in terms of size and power. For both level and power comparisons we considered the significance level $\alpha = 0.10, 0.05$ and 0.01. Empirical levels and power were independent of the values of location parameters chosen so we took $(u_1, u_2) = (0.1, 0.5)$ for $L = 2$. The common scale parameter was taken as 0.33. Each experiment for empirical levels was based on 2000 replications. Asymptotic null distribution of the statistic EP is not known and hence it was not included in the computation of empirical levels. For power comparison we first calculated empirical critical values for all the statistics by simulation. Each critical value was based on 10,000 replications and power was based on 2000 replications. Empirical levels and power, for $L =2$; $\alpha = 0.10, 0.05, 0.01$, are reported in Tables 5.13, 5.14 and 5.15. Simulations, in this section, have been studied only for $L =2$ groups. This was partly because of computational costs and partly because similar conclusions are expected, for other values of L.

Results

From Tables 5.13, 5.14, 5.15, we can see that the likelihood ratio statistic LRb is, in general, liberal. The $C(\alpha)$ statistic CLb holds nominal level well for complete samples except for $\alpha = 0.01$ and small samples, in which case it is conservative. For censored sample situations this statistic is always conservative, particularly as the number of failures in the groups decreases. The marginal likelihood ratio statistic ML and the modified likelihood ratio statistic MB hold nominal level well, except in very small sample situations such as $n_1 = r_1 = n_2 = r_2 = 5$, where the statistic MB is slightly liberal. This behaviour is similar for all values of α, although some inconsistent behaviour for the statistic MB is evident at $\alpha = 0.01$. Note that two times the standard error of the probabilities reported is roughly 0.005, 0.010 and 0.013 respectively, for $\alpha = 0.01$, 0.05 and 0.10.

Now, we compare the power performance of all five statistics. For equal sample size situations all statistics have similar power. For the situation in which sample sizes in the groups are different, power of the statistics MB, ML and CLb is in general similar except in some instances, in which CLb is more powerful, the statistic EP has the least power. Since the statistic ML, based on its asymptotic chi- square distribution holds the levels well and when all the statistics hold the level, the statistic ML performs reasonably well in terms of power, we recommend its use in practice. Notice that the statistic CLb is in general conservative, but it has slight power advantage in some instances. For this slight advantage in power, we find that it may not be worthwhile to try to improve its level.

5.8 EXAMPLES

Example 1: The data analyzed by Lawless and Mann (1976) and Lawless (1982, Table 4.3.2) refer to time (T) to breakdown of a particular type of insulating fluid, subject to constant voltage stress. The experiment was conducted at 7 different voltage levels. The time T is assumed to follow a Weibull distribution and hence logT has extreme value distribution with parameters u and b depending on voltage levels. The Weibull data (time in minutes, voltage in kilovolts) are as follows:

Voltage KV	Sample size n_i	Breakdown times T_{ij}
26	3	5.79, 1579.52, 2323.7
28	5	68.85, 426.07, 110.29, 108.29, 1067.6
30	11	17.05, 22.66, 21.02, 175.88, 139.07, 144.12, 20.46, 43.40, 104.9, 47.3, 7.74
32	15	0.40, 82.85, 9.88, 89.29, 215.10, 2.75, 0.79, 15.93, 3.91, 0.27, 0.69, 100.58, 27.80, 13.95, 53.24
34	19	0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89
36	15	1.97, 0.59, 2.58, 1.69, 2.71, 25.50, 0.35, 0.99, 3.99, 3.67, 2.07, 0.96, 5.35, 2.90, 13.77
38	8	0.47, 0.73, 1.40, 0.74, 0.39, 1.13, 0.09, 2.38

We first test the equality of the scale parameters. The value of the recommended statistic ML is 8.35 with 6 degrees of freedom. It appears that there is no evidence against the assumption of a common scale parameter at 5% level. The recommended statistic for testing the equality of the location parameters in the presence of a common scale parameter is ACLu, which is approximately distributed as $\chi^2(d)$. The value of c = 0.88;

d = 6.81; CLu = 66.93; ACLu = 75.93 and $\chi^2_{0.05}(6.81) = 18.16$. The p-values for CLu and ACLu are, for all practical purposes, indistinguishable from zero. This indicates strong evidence that the location parameters are not the same for all voltage levels.

Example 2: The following data are taken from Lawless (1982, p. 202). They refer to failure times (in hours) for two types of polyethylene cable insulation, obtained from an accelerated life test. Of the 10 items of each type tested, 9 of each failed.

Type I: 5.1, 9.2, 9.3, 11.8, 17.7, 19.4, 22.1, 26.7, 37.3

Type II: 11.0, 15.1, 18.3, 24.0, 29.1, 38.6, 44.2, 45.1, 50.9.

Assuming that the failure times for each type follow the Weibull distribution, we obtain, for testing the homogeneity of scale parameters. ML = 0.43, which is distributed as chi-square with 1 degree of freedom. This shows that there is a strong evidence in favour of a common b. For testing the equality of location parameters with a common scale parameter, we obtain c = 0.73, d = 1.38, CLu = 2.83, ACLu = 3.8, $\chi^2_{0.10}(1.38) = 3.48$, $\chi^2_{0.05}(1.38) = 4.72$ and $\chi^2_{0.01}(1.38) = 7.70$. The p-values for the statistics CLu and ACLu are 0.08 and 0.09 respectively. Thus, there is no evidence at 5% level against the hypothesis that the location parameters are same.

5.9 APPENDIX

Expected Mixed Partial Derivatives for the Derivation of the C(α) Statistics for Homogeneity Testing in Several Extreme Value Distributions

Denote that $Z_{ij} = (X_{ij} - u)/b$, for $j = 1,\ldots,n_i$; $i=1,\ldots,L$. Then, for complete samples the expected mixed partial derivatives are

$$D_{ii} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i^2}\right|_{H_0}\right) = \frac{1}{b^2}\,E\sum_{j=1}^{n_i}\exp\left(Z_{ij}\right) = \frac{n_i}{b^2},$$

$$D_{ij} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial \phi_j}\right|_{H_0}\right) = 0,$$

$$A_{i1} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial u}\right|_{H_0}\right) = \frac{1}{b^2}\,E\sum_{j=1}^{n_i}\exp\left(Z_{ij}\right) = \frac{n_i}{b^2},$$

$$A_{i2} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial b}\right|_{H_0}\right) = \frac{1}{b^2}\,E\sum_{j=1}^{n_i}\left\{(Z_{ij}+1)\exp(Z_{ij}) - 1\right\} = \frac{n_i(1-\gamma)}{b^2},$$

$$B_{11} = -E\left(\left.\frac{\partial^2 l}{\partial u^2}\right|_{H_0}\right) = \frac{1}{b^2}\,E\sum_{i=1}^{L}\sum_{j=1}^{n_i}\exp(Z_{ij}) = \frac{N}{b^2}, \quad N = \sum_{i=1}^{L} n_i,$$

$$B_{21} = B_{12} = -E\left(\left. \frac{\partial^2 l}{\partial u \partial b} \right|_{H_0} \right) = \frac{1}{b^2} E \sum_{i=1}^{L} \sum_{j=1}^{n_i} \left\{ (Z_{ij}+1) \exp(Z_{ij}) - 1 \right\}$$

$$= \frac{N\,(1-\gamma)}{b^2},$$

$$B_{22} = -E\left(\left. \frac{\partial^2 l}{\partial b^2} \right|_{H_0} \right) = \frac{1}{b^2} E \sum_{i=1}^{L} \sum_{j=1}^{n_i} \left\{ (Z_{ij}^2+2Z_{ij}) \exp(Z_{ij}) - 2Z_{ij} - 1 \right\}$$

$$= \frac{N\,K}{b^2},$$

where $K = \pi^2/6 + (1-\gamma)^2$. For k = 1,...,L, and i = 1,...,(L-1), we define $Z_{ij} = (X_{ij} - u_i)/b$, j = 1,...,$n_j$. Thus we have

$$C_{ii} = -E\left(\left. \frac{\partial^2 l}{\partial \phi_i^2} \right|_{H_1} \right) = \frac{1}{b^2} E \sum_{j=1}^{n_i} \left\{ (Z_{ij}^2+Z_{ij}) \exp(Z_{ij}) - 2Z_{ij} - 1 \right\}$$

$$= \frac{n_i\,K}{b^2},$$

$$C_{ij} = -E\left(\left. \frac{\partial^2 l}{\partial \phi_i \partial \phi_i} \right|_{H_1} \right) = 0 \quad , \quad i \neq j \, ,$$

$$E_{ii} = -E\left(\left. \frac{\partial^2 l}{\partial\phi_i \partial u_i} \right|_{H_1} \right) = \frac{1}{b^2} E \sum_{j=1}^{n_i} \left\{ (Z_{ij}+1)\exp(Z_{ij}) - 1 \right\} = \frac{n_i\,(1-\gamma)}{b^2},$$

$$E_{ik} = -E\left(\left. \frac{\partial^2 l}{\partial\phi_i \partial u_k} \right|_{H_1} \right) = 0, \quad k \neq i,$$

$$E_{iL+1} = -E\left(\left. \frac{\partial^2 l}{\partial\phi i \partial b} \right|_{H_1} \right) = \frac{1}{b^2} E \sum_{j=1}^{n_i} \left\{ (Z_{ij}^2+Z_{ij})\exp(Z_{ij}) - 2Z_{ij} - 1 \right\}$$

$$= \frac{n_i\,K}{b^2},$$

$$F_{kk} = -E\left(\left. \frac{\partial^2 l}{\partial u_k^2} \right|_{H_1} \right) = \frac{1}{b^2} E \sum_{j=1}^{n_k} \exp(Z_{kj}) = \frac{n_k}{b^2},$$

$$F_{kk'} = -E\left(\left. \frac{\partial^2 l}{\partial u_k \partial u_k'} \right|_{H_1} \right) = 0, \quad k \neq k',$$

$$F_{L+1,k} = F_{k,L+1} = -E\left(\frac{\partial^2 l}{\partial u_k \partial b} \Bigg|_{H_1} \right) = \frac{1}{b^2}\, E \sum_{j=1}^{n_k} \left\{ (Z_{kj}+1)\exp(Z_{kj}) - 1 \right\}$$

$$= \frac{n_k\,(1-\gamma)}{b^2},$$

and

$$F_{L+1,L+1} = -E\left(\frac{\partial^2 l}{\partial b^2} \Bigg|_{H_1} \right) = \frac{1}{b^2} \sum_{i=1}^{L} \sum_{j=1}^{n_i} \left\{ (Z_{ij}^2+Z_{ij})\exp(Z_{ij}) - 2Z_{ij} - 1 \right\}$$

$$= \frac{N\,K}{b^2}.$$

For censored samples recall that

$$\sum_{j=1}^{*} t_{ij} = \sum_{j=1}^{r_i} t_{ij} + (n_i - r_i)\, t_{ir_i}.$$

Now, we denote $C_{ij} = n_i! / \{(j-1)!\,(n_i-j)!\}$ and we define

$$I_{1i} = \sum_{j=1}^{r_i} c_{ij} \sum_{s=1}^{j} (-1)^{s-1} \binom{j-1}{s-1} \frac{1}{(n_i-j+s)^2},$$

$$I_{2i} = (n_i-r_i)\, c_{ir_i} \sum_{s=1}^{r_i} (-1)^{s-1} \binom{r_i-1}{s-1} \frac{1}{(n_i-r_i+s)^2},$$

$$I_{3i} = \sum_{j=1}^{r_i} c_{ij} \sum_{s=1}^{j} (-1)^{s-1} \binom{j-1}{s-1} \left\{ \frac{2-\gamma-\log(n_i-j+s)}{(n_i-j+s)^2} \right\},$$

$$I_{4i} = (n_i-r_i)\, c_{ir_i} \sum_{s=1}^{r_i} (-1)^{s-1} \binom{r_i-1}{s-1} \left\{ \frac{2-\gamma-\log(n_i-r_i+s)}{(n_i-r_i+s)^2} \right\},$$

$$I_{5i} = \sum_{j=1}^{r_i} c_{ij} \sum_{s=1}^{j} (-1)^{s-1} \binom{j-1}{s-1} \left\{ (\pi^2/6-2) + \frac{(2-\gamma-\log(n_i-j+s))^2}{(n_i-j+s)^2} \right\},$$

$$I_{6i} = (n_i-r_i)\, c_{ir_i} \sum_{j=1}^{r_i} (-1)^{s-1} \binom{r_i-1}{s-1} \left[\frac{(\pi^2/6-2) + (2-\gamma-\log(n_i-r_i+s))^2}{(n_i-r_i+s)^2} \right\},$$

and

$$I_{7i} = \sum_{j=1}^{r_i} c_{ij} \sum_{s=1}^{j} (-1)^{s-1} \binom{j-1}{s-1} \left\{ \frac{\gamma+\log(n_i-j+s)}{n_i-j+s} \right\}.$$

Then the expected mixed partial derivatives for censored samples are

$$D_{ii} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i^2}\right|_{H_0}\right) = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial u}\right|_{H_0}\right) = A_{i1}$$

$$= \frac{1}{b^2} E\left\{\sum_{j=1}^{*} \exp\,(Z_{ij})\right\} = \frac{I_{1i} + I_{2i}}{b^2},$$

$$D_{ij} = -E\left(\left.\frac{\partial^2 l}{\partial \phi i \partial \phi j}\right|_{H_0}\right) = 0, \qquad i \neq j,$$

$$A_{i2} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial b}\right|_{H_0}\right) = \frac{1}{b^2} E\left\{\sum_{j=1}^{*} (Z_{ij}+1)\exp(Z_{ij}) - r_i\right\}$$

$$= \frac{(I_{3i} + I_{4i} - r_i)}{b^2},$$

$$B_{11} = -E\left(\left.\frac{\partial^2 l}{\partial u^2}\right|_{H_0}\right) = \frac{1}{b^2} E\left\{\sum_{i=1}^{L} \sum_{j=1}^{*} \exp(Z_{ij})\right\}$$

$$= \frac{1}{b^2} \sum_{i=1}^{L} (I_{1i} + I_{2i}),$$

$$B_{21} = B_{12} = -E\left(\left.\frac{\partial^2 l}{\partial u \partial b}\right|_{H_0}\right)$$

$$= \frac{1}{b^2}\sum_{i=1}^{L} E\left[\sum_{j=1}^{*}(Z_{ij}+1)\exp(Z_{ij}) - r_i\right]$$

$$= \frac{1}{b^2}\sum_{i=1}^{L}(I_{3i} + I_{4i} - r_i),$$

and

$$B_{22} = -E\left(\left.\frac{\partial^2 l}{\partial b^2}\right|_{H_0}\right)$$

$$= \frac{1}{b^2}\sum_{i=1}^{L} E\left[\sum_{j=1}^{*}(Z_{ij}^2+2Z_{ij})\exp(Z_{ij}) - 2\sum_{j=1}^{r_i} Z_{ij} - r_i\right]$$

$$= \frac{1}{b^2}\sum_{i=1}^{L}(I_{5i} + I_{6i} - 2I_{7i} - r_i).$$

Now, we take $Z_{ij} = (X_{ij} - u_i)/b$, $j = 1,\ldots,n_i$, $i = 1,\ldots,L$. Then

$$C_{ii} = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i^2}\right|_{H_1}\right) = -E\left(\left.\frac{\partial^2 l}{\partial \phi_i \partial b}\right|_{H_1}\right) = E_{i,L+1}$$

$$= \frac{1}{b^2} E\left[\sum_{j=1}^{*}(Z_{ij}^2+2Z_{ij})\exp(Z_{ij}) - 2\sum_{i=1}^{r_i} Z_{ij} - r_i\right]$$

$$= \frac{1}{b^2}(I_{5i} + I_{6i} + 2I_{7i} - r_i),$$

$$C_{ij} = -E\left(\left.\frac{\partial^2 l}{\partial\phi_i \partial\phi_j}\right|_{H_1}\right) = 0, \quad i \neq j$$

$$E_{ii} = -E\left(\left.\frac{\partial^2 l}{\partial\phi_i \partial u_i}\right|_{H_1}\right) = \frac{1}{b^2}\, E\left[\sum_{j=1}^{*} (Z_{ij}+1)\exp(Z_{ij}) - r_i\right]$$

$$= \frac{1}{b^2}\,(I_{3i} + I_{4i} - r_i),$$

$$E_{ik} = -E\left(\left.\frac{\partial^2 l}{\partial\phi_i\, \partial u_k}\right|_{H_i}\right) = 0 \quad , \quad k \neq i \quad ,$$

$$E_{i,L+1} = -E\left(\left.\frac{\partial^2 l}{\partial\phi_i \partial b}\right|_{H_1}\right)$$

$$= \frac{1}{b^2}\, E\left[\sum_{j=1}^{*} (Z_{ij}^2 + 2Z_{ij})\exp(Z_{ij}) - 2\sum_{j=1}^{r_i} Z_{ij} - r_i\right]$$

$$= \frac{1}{b^2}\,(I_{5i} + I_{6i} - 2I_{7i} - r_i),$$

$$F_{kk} = -E\left(\left.\frac{\partial^2 l}{\partial u_k^2}\right|_{H_1}\right) = \frac{1}{b^2} E\left[\sum_{j=1}^{*} \exp(Z_{kj})\right]$$

$$= \frac{1}{b^2}(I_{1k} + I_{2k}),$$

$$F_{kk'} = -E\left(\left.\frac{\partial^2 l}{\partial u_k \partial u_k'}\right|_{H_1}\right) = 0\,, \qquad k \neq k',$$

$$F_{k,L+1} = F_{L+1,k} = -E\left(\left.\frac{\partial^2 l}{\partial u_k \partial b}\right|_{H_1}\right)$$

$$= \frac{1}{b^2} E\left[\sum_{j=1}^{*} (Z_{kj}+1)\exp(Z_{kj}) - r_k\right]$$

$$= \frac{1}{b^2}(I_{3k} + I_{4k} - r_k),$$

and

$$F_{L+1,L+1} = -E\left(\left.\frac{\partial^2 l}{\partial b^2}\right|_{H_1}\right)$$

$$= \frac{1}{b^2}\sum_{i=1}^{L} E\left[\sum_{j=1}^{*} (Z_{ij}^2+2Z_{ij})\exp(Z_{ij}) - 2\sum_{j=1}^{r_i} Z_{ij} - r_i\right]$$

$$= \frac{1}{b^2}\sum_{i=1}^{L}(I_{5i} + I_{6i} - 2I_{7i} - r_i).$$

The above formulae for the expected negative mixed partial derivatives are mathematically and computationally messy. Simple but very accurate formualae can be obtained by using section 2.14.

Now, we define

$$t_{jn_i} = \sum_{s=1}^{j} \frac{1}{n_i-s+1} \; and \; d_{jn_i} = \sum_{s=1}^{j} \frac{1}{(n_i-s+1)^2}.$$

For notational simplicity we denote t_{jn_i} = t_j and d_{jn_i} = d_j; that is, t_j and d_j will be

understood to be with respect to n_i. Then we have for i = 1,...,(L-1),

$$D_{ii} = A_{i1} = \frac{r_i}{b^2}, \quad D_{ij} = 0 \;, \quad i \neq j,$$

$$A_{i2} = \frac{1}{b^2} \sum_{j=1}^{*} (t_j \log t_j + \frac{d_j}{2t_j}), \quad ,$$

$$B_{11} = \frac{1}{b^2} \sum_{j=1}^{L} r_i$$

$$B_{12} = B_{21} = \frac{1}{b^2} \sum_{i=1}^{L} \sum_{j=1}^{*} (t_j \log t_j + \frac{d_j}{2t_j}) \;,$$

$$B_{22} = \frac{1}{b^2} \sum_{i=1}^{L} \left[\sum_{j=1}^{*} (2+\log t_j)(t_j \log t_j + \frac{d_j}{t_j}) - \sum_{j=1}^{r_i} (2 \log t_j - \frac{d_j}{t_j^2}) - r_i \right],$$

$$C_{ii} = \frac{1}{b^2} \left[\sum_{j=1}^{*} (2+\log t_j)(t_j \log t_j + \frac{d_j}{t_j}) - \sum_{j=1}^{r_i} (2 \log t_j - \frac{d_j}{t_j^2}) - r_i \right]$$

$C_{ij} = 0, \; i \neq j,$

$$E_{ii} = \frac{1}{b^2} \sum_{j=1}^{*} (t_j \log t_j + \frac{d_j}{2t_j}),$$

$E_{ik} = 0, \; i \neq k,$

$$E_{i,L+1} = \frac{1}{b^2} \left[\sum_{j=1}^{*} (2 + \log t_j)(t_j \log t_j + \frac{d_j}{t_j}) - \sum_{j=1}^{r_i} (2 \log t_j - \frac{d_j}{t_j^2}) - r_i \right]$$

$F_{kk} = r_k/b^2,$

$F_{kk'} = 0, \; k \neq k',$

$$F_{k,L+1} = \frac{1}{b^2} \sum_{j=1}^{*} (t_j \log t_j + \frac{d_j}{2\, t_j})$$

and

$$F_{L+1,L+1} = \frac{1}{b^2} \sum_{i=1}^{L} \left[\sum_{j=1}^{*} (2+\log t_j)(t_j \log t_j + \frac{d_j}{t_j}) - \sum_{j=1}^{r_i} (2 \log t_j - \frac{d_j}{t_j^2}) - r_i \right]$$

Note that the approximations are involved only in two quantities, namely,

$$I_i = \sum_{j=1}^{*} (t_j \log t_j + \frac{d_j}{2t_j})$$

and

$$J_i = \sum_{j=1}^{*} (2+\log t_j) (t_j \log t_j + \frac{d_j}{t_j}) - \sum_{j=1}^{r_i} (2 \log t_j - \frac{d_j}{t_j^2} - r_i \, .$$

We compare in chapter 7 these values with the corresponding exact values for a single sample for some combinations of (n,r). For a single sample we denote the above values, in chapter 7, as C and J.

Table 5.1. Empirical level (%) of the Test Statistics LRu, and $CMLu$ based on 2000 replications

		$L=2$				$L=3$				$L=5$		
		α				α				α		
Tests	n_1,r_1 n_2,r_2	.10	.05	.01	n_1,r_1 n_2,r_2 n_3,r_3	.10	.05	.01	n_1,r_1,n_2,r_2,n_3,r_3 n_4,r_4,n_5,r_5	.10	.05	.01
LRu	5,5	17.0	10.6	3.3	5,5	17.8	11.0	3.6	5,5,5,5,5,5	18.1	10.3	2.7
$CMLu$		10.9	3.2	0.0	5,5	9.7	3.2	0.2		7.6	2.8	0.4
LRu	10,10	13.1	7.2	2.0	10,10	13.3	7.6	2.4	10,10,10,10,10,10	13.5	7.1	2.4
$CMLu$		10.1	4.3	0.3	10,10	9.4	4.5	0.1		8.6	3.9	0.5
LRu	10,5	17.5	10.9	3.8	10,5	19.3	11.0	3.2	10,5,10,5,10,5	20.8	13.3	4.0
$CMLu$		9.4	2.8	0.0	10,5	6.7	2.1	0.3		7.6	2.6	0.2
LRu	10,3	26.2	17.6	5.8	10,3	28.1	19.4	7.5	10,3,10,3,10,3	33.1	21.5	8.4
$CMLu$		6.7	0.2	0.0	10,3	6.4	1.2	0.0		6.2	2.1	0.2
LRu	20,20	12.1	5.8	1.3	20,20	12.4	6.9	1.9	20,20,20,20,20,20	11.9	6.8	1.3
$CMLu$		10.5	4.5	0.7	20,20	10.6	5.0	0.1		9.4	4.6	0.8
LRu	20,10	13.1	8.0	1.8	20,10	15.3	9.0	2.3	20,10,20,10,20,10	14.9	8.4	2.0
$CMLu$		10.0	4.2	0.1	20,10	9.6	4.0	0.3		8.4	3.2	0.5
LRu	20,5	18.8	10.7	3.1	20,5	20.7	12.1	3.3	20,5,20,5,20,5	22.4	13.0	4.3
$CMLu$		9.4	2.1	0.0	20,5	7.3	1.7	0.0		6.9	2.3	0.3
LRu	20,20	12.1	5.9	1.3	20,20	12.0	6.6	1.7	20,20,16,16,12,12	13.5	7.0	1.7
$CMLu$		9.9	3.9	0.3	12,12	9.5	4.7	0.7		9.7	4.8	1.0
LRu	20,10	14.4	8.3	2.4	20,10	15.7	9.1	2.4	20,10,16,8,12,6	17.8	10.3	3.1
$CMLu$		9.1	3.3	0.5	12,6	8.5	3.4	0.4		8.1	3.5	0.5
LRu	20,5	20.4	13.0	3.8	20,5	22.1	14.6	6.7	20,5,16,4,12,3	28.8	19.1	7.0
$CMLu$		7.4	1.7	0.0	12,3	7.2	2.3	0.0		6.4	2.7	0.2

Table 5.2: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.01$; $L = 2$.

		b = 0.3				b = 1.5			
		(u_1,u_2)				(u_1,u_2)			
Tests	n_1,r_1 n_2,r_2	(0,0.1)	(0,0.2)	(0,0.5)	(0,1.0)	(0,0.1)	(0,0.2)	(0,0.5)	(0,1.0)
LRu	5,5	1.55	3.90	26.90	85.20	1.40	1.30	1.55	3.90
SP	5,5	1.40	2.95	23.25	82.50	1.00	1.10	1.40	2.95
CLu		1.55	3.60	26.40	83.20	1.30	1.15	1.55	3.60
LRu	5,3	1.25	1.75	7.15	31.10	1.20	1.20	1.25	1.75
SP	5,3	1.15	1.65	6.50	29.50	1.15	1.15	1.15	1.65
CLu		1.25	1.95	7.05	30.40	1.15	1.15	1.25	1.95
LRu	10,10	2.70	10.15	72.75	99.95	1.35	1.35	2.70	10.15
SP	10,10	2.55	9.20	71.15	99.95	1.05	1.40	2.55	9.20
CLu		2.75	9.55	72.40	99.95	1.30	1.25	2.75	9.55
LRu	10,7	1.75	4.70	41.70	97.60	1.15	1.20	1.75	4.70
SP	10,7	1.65	4.95	40.65	97.75	1.25	1.25	1.65	4.95
CLu		1.75	4.65	41.85	97.50	1.15	1.10	1.75	4.65
LRu	10,5	1.20	2.70	19.90	77.10	1.25	0.90	1.20	2.70
SP	10,5	1.40	2.80	20.15	77.85	1.10	1.00	1.40	2.80
CLu		1.30	2.95	20.10	77.00	1.15	1.05	1.30	2.95
LRu	10,3	1.00	1.75	6.80	28.70	0.75	0.70	1.00	1.75
SP	10,3	1.05	1.80	6.70	28.70	0.80	0.70	1.05	1.80
CLu		0.95	1.65	6.25	27.90	0.80	0.85	0.95	1.65
LRu	20,20	5.95	27.65	98.85	100.	1.05	1.65	5.95	27.65
SP	20,20	5.50	25.75	98.40	100.	1.25	1.50	5.50	25.75
CLu		5.70	27.75	98.90	100.	0.95	1.70	5.70	27.75
LRu	20,15	4.65	18.90	94.10	100.	1.10	1.65	4.65	18.90
SP	20,15	4.65	18.85	94.05	100.	1.10	1.75	4.65	18.85
CLu		4.55	19.05	93.55	100.	1.00	1.50	4.55	19.05
LRu	20,10	2.50	9.20	68.25	99.95	0.80	0.90	2.50	9.20
SP	20,10	2.65	9.60	69.70	99.95	0.90	1.05	2.65	9.60
CLu		2.50	9.15	67.60	99.90	0.90	0.95	2.50	9.15

Table 5.2 continued

LRu	20,5	1.50	3.30	20.85	79.80	1.00	1.10	1.50	3.30
SP	20,5	1.40	3.15	20.95	79.70	1.00	1.00	1.40	3.15
CLu		1.50	3.30	21.10	79.65	0.95	1.05	1.50	3.30
LRu	20,20	3.15	18.25	92.05	100.	0.65	1.10	3.15	18.25
SP	10,10	3.55	19.60	91.75	100.	0.75	1.25	3.55	19.60
CLu		7.05	26.75	95.40	100.	1.25	1.85	7.05	26.75
LRu	20,15	2.90	11.05	76.80	100.	0.85	1.00	2.90	11.05
SP	10,7	0.10	0.45	26.85	98.00	0.85	0.85	0.10	0.45
CLu		4.90	17.85	85.50	100.	1.30	1.85	4.90	17.85
LRu	20,10	1.65	4.95	44.50	98.15	1.00	1.00	1.65	4.95
SP	10,5	1.95	5.80	48.90	98.60	0.95	1.05	1.95	5.80
CLu		3.45	9.35	60.65	99.30	1.30	1.70	3.45	9.35
LRu	20,5	1.45	2.70	12.70	53.50	0.95	1.20	1.45	2.70
SP	10,3	1.70	3.10	15.50	59.50	1.05	1.10	1.70	3.10
CLu		2.35	4.60	20.85	70.55	1.15	1.50	2.35	4.60

Table 5.3: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.10$; L = 2.

		b = 0.3				b = 1.5			
		(u_1,u_2)				(u_1,u_2)			
Tests	n_1,r_1 n_2,r_2	(0,0.1)	(0,0.2)	(0,0.5)	(0,1.0)	(0,0.1)	(0,0.2)	(0,0.5)	(0,1.0)
LRu	5,5	7.35	14.05	56.75	98.05	5.45	5.75	7.35	14.05
SP	5,5	7.60	14.20	56.40	98.05	5.55	5.75	7.60	14.20
CLu		7.40	13.75	56.75	97.60	5.45	5.50	7.40	13.75
LRu	5,3	6.35	9.75	27.40	69.45	4.50	4.70	6.35	9.75
SP	5,3	6.30	9.45	27.75	69.30	4.70	4.65	6.30	9.45
CLu		6.05	9.40	26.65	68.55	4.40	4.60	6.05	9.40
LRu	10,10	10.55	26.95	91.80	100.	5.35	6.00	10.55	26.95
SP	10,10	10.25	27.20	90.65	100.	5.05	5.95	10.25	27.20
CLu		10.70	27.05	91.75	100.	5.35	6.00	10.70	27.05
LRu	10,7	8.80	18.20	74.10	99.80	5.35	6.00	8.80	18.20
SP	10,7	8.40	18.10	74.30	99.95	5.45	6.10	8.40	18.10
CLu		8.55	18.60	74.05	99.80	5.50	6.00	8.55	18.60
LRu	10,5	6.90	12.30	52.10	97.10	5.15	5.35	6.90	12.30
SP	10,5	7.20	12.70	51.85	97.10	5.35	5.35	7.20	12.70
CLu		7.00	12.35	52.30	97.00	5.30	5.50	7.00	12.35
LRu	10,3	5.35	7.85	24.60	68.25	4.25	4.30	5.35	7.85
SP	10,3	5.15	7.55	24.15	67.45	4.20	4.30	5.15	7.55
CLu		5.20	7.60	24.50	67.90	4.35	4.50	5.20	7.60
LRu	20,20	17.00	52.15	100.	100.	5.00	6.80	17.00	52.15
SP	20,20	16.65	51.00	99.95	100.	5.05	6.70	16.65	51.00
CLu		17.15	52.10	99.95	100.	5.10	6.55	17.15	52.10
LRu	20,15	13.75	39.60	99.05	100.	5.10	6.60	13.75	39.60
SP	20,15	13.05	38.50	99.00	100.	5.10	6.40	13.05	38.50
CLu		13.95	39.60	99.10	100.	5.10	6.50	13.95	39.60
LRu	20,10	10.30	24.30	89.95	100.	5.10	5.80	10.30	24.30
SP	20,10	10.15	24.60	90.15	100.	4.95	5.55	10.15	24.60
CLu		10.15	24.20	89.95	100.	5.15	5.80	10.15	24.20

Table 5.3 continued

LRu	20,5	6.10	11.60	52.20	96.95	4.60	4.70	6.10	11.60
SP	20,5	6.10	11.60	51.85	96.65	4.60	4.65	6.10	11.60
CLu		6.10	11.60	51.60	96.95	4.45	4.75	6.10	11.60
LRu	20,20	14.55	39.40	98.00	100.	5.10	5.85	14.55	39.40
SP	10,10	14.00	37.75	97.85	100.	4.95	6.15	14.00	37.75
CLu		18.05	45.60	98.85	100.	5.90	8.05	18.05	45.60
LRu	20,15	9.90	28.60	91.60	100.	5.45	5.60	9.90	28.60
SP	10,7	8.95	25.85	90.15	100.	5.10	5.70	8.95	25.85
CLu		14.70	37.90	95.55	100.	5.95	7.50	14.70	37.90
LRu	20,10	7.10	16.15	74.35	99.70	4.85	5.00	7.10	16.15
SP	10,5	7.60	17.55	75.40	99.80	4.80	5.20	7.60	17.55
CLu		10.75	26.80	83.55	99.95	5.55	6.75	10.75	26.80
LRu	20,5	6.15	10.45	36.35	86.10	4.65	4.80	6.15	10.45
SP	10,3	7.35	12.25	42.05	89.85	4.85	5.50	7.35	12.25
CLu		8.35	14.25	48.15	93.15	4.90	5.50	8.35	14.25

Table 5.4: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.10$; $L = 2$.

		b = 0.3				b = 1.5			
		(u_1,u_2)				(u_1,u_2)			
Tests	n_1,r_1 n_2,r_2	(0,0.1)	(0,0.2)	(0,0.5)	(0,1.0)	(0,0.1)	(0,0.2)	(0,0.5)	(0,1.0)
LRu	5,5	14.80	25.25	71.95	99.50	10.80	10.90	14.80	25.25
SP	5,5	14.35	23.95	70.75	99.45	10.70	11.45	14.35	23.95
CLu		14.55	24.85	71.40	99.25	10.75	11.20	14.55	24.85
LRu	5,3	12.95	17.05	43.35	86.90	10.35	10.80	12.95	17.05
SP	5,3	12.75	17.15	43.25	86.55	10.35	10.80	12.75	17.15
CLu		13.15	16.90	43.10	86.30	10.10	10.40	13.15	16.90
LRu	10,10	18.75	39.30	95.75	100.	10.45	11.30	18.75	39.30
SP	10,10	18.40	39.35	95.65	100.	10.40	11.35	18.40	39.35
CLu		18.50	39.40	95.80	100.	10.40	11.30	18.50	39.40
LRu	10,7	14.70	28.70	85.55	100.	10.60	11.35	14.70	28.70
SP	10,7	14.60	28.10	85.15	100.	11.00	11.00	14.60	28.10
CLu		15.10	28.65	85.95	100.	10.75	11.45	15.10	28.65
LRu	10,5	13.10	22.80	69.60	99.45	10.15	10.85	13.10	22.80
SP	10,5	13.00	22.80	69.80	99.50	10.50	11.25	13.00	22.80
CLu		13.25	22.70	69.45	99.45	10.15	10.60	13.25	22.70
LRu	10,3	11.45	15.45	40.90	86.00	9.60	9.90	11.45	15.45
SP	10,3	11.35	15.50	41.25	85.55	9.50	9.85	11.35	15.50
CLu		11.15	15.25	41.00	86.00	9.50	10.10	11.15	15.25
LRu	20,20	26.55	64.85	100.	100.	10.15	12.30	26.55	64.85
SP	20,20	25.80	62.75	100.	100.	10.35	12.55	25.80	62.75
CLu		26.65	64.75	100.	100.	10.30	12.30	26.65	64.75
LRu	20,15	21.85	52.25	99.65	100.	10.45	11.65	21.85	52.25
SP	20,15	21.80	52.25	99.65	100.	10.35	11.70	21.80	52.25
CLu		21.70	52.20	99.70	100.	10.40	11.45	21.70	52.20
LRu	20,10	17.35	37.45	96.00	100.	9.85	10.75	17.35	37.45
SP	20,10	17.40	37.45	95.85	100.	9.95	11.05	17.40	37.45
CLu		17.20	37.60	96.00	100.	9.65	10.80	17.20	37.60

Table 5.4 continued

LRu	20,5	11.70	21.25	69.70	98.70	9.45	9.75	11.70	21.25
SP	20,5	11.80	21.15	69.70	98.75	9.30	9.45	11.80	21.15
CLu		11.50	21.50	69.70	98.75	9.50	9.75	11.50	21.50
LRu	20,20	23.10	50.60	99.20	100.	10.55	12.55	23.10	50.60
SP	10,10	23.05	49.15	99.10	100.	10.25	12.00	23.05	49.15
CLu		26.35	54.80	99.40	100.	11.35	13.95	26.35	54.80
LRu	20,15	17.95	40.60	96.15	100.	10.30	11.00	17.95	40.60
SP	10,7	16.65	37.95	95.30	100.	10.15	10.80	16.65	37.95
CLu		22.95	47.50	97.30	100.	11.70	13.50	22.95	47.50
LRu	20,10	14.40	29.70	85.50	100.	10.40	10.65	14.40	29.70
SP	10,5	14.95	31.20	86.35	100.	10.25	10.50	14.95	31.20
CLu		17.85	37.05	91.[illegible]	100.	10.35	11.20	17.85	37.05
LRu	20,3	11.45	17.25	53.95	95.10	10.05	10.20	11.45	17.25
SP	10,3	13.05	20.30	59.35	96.45	9.75	10.10	13.05	20.30
CLu		14.45	22.25	[illegible]	97.10	9.30	10.30	14.45	22.25

Table 5.5: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.01$, b = 0.3, 1.5; L = 3.

		b = 0.3				b = 1.5			
		(u_1,u_2,u_3)				(u_1,u_2,u_3)			
Tests	n_1,r_1 n_2,n_2 n_3,r_3	(0,0,0)	(0,.1,.2)	(0,.2,.4)	(0,.5,1)	(0,0,0)	(0,.1,.2)	(0,.2,.4)	(0,.5,1)
LRu	5,5	1.25	2.65	10.35	72.30	1.25	1.30	1.60	3.30
SP	5,5	1.10	2.55	9.60	69.80	1.10	1.35	1.15	3.00
CLu	5,5	0.70	2.70	8.80	69.20	0.70	0.80	1.00	2.95
LRu	5,3	1.05	1.40	2.80	21.90	1.05	1.05	1.15	1.30
SP	5,3	1.10	1.35	2.80	20.90	1.10	1.05	1.00	1.35
CLu	5,3	1.05	1.90	3.85	20.95	1.05	1.20	1.25	2.00
LRu	10,10	1.05	5.45	29.95	99.45	1.05	1.10	1.55	6.85
SP	10,10	1.10	5.75	29.25	99.35	1.10	1.00	1.55	7.25
CLu	10,10	1.30	5.05	29.25	98.80	1.30	1.40	2.00	6.65
LRu	10,7	1.10	3.55	15.05	92.35	1.10	1.20	1.45	4.40
SP	10,7	1.05	3.75	15.95	91.90	1.05	1.35	1.65	4.35
CLu	10,7	0.95	3.40	14.70	85.95	0.95	1.00	1.40	4.10
LRu	10,5	1.05	1.75	6.85	65.10	1.05	1.15	1.95	9.15
SP	10,5	0.95	1.60	7.10	65.40	0.95	1.05	1.10	2.05
CLu	10,5	1.00	1.80	8.60	58.65	1.00	0.60	0.70	2.15
LRu	10,3	1.00	1.55	3.50	21.50	1.00	1.10	1.05	1.60
SP	10,3	1.05	1.65	3.60	22.25	1.05	1.20	1.20	1.85
CLu	10,3	1.35	2.60	5.50	22.10	1.35	1.40	1.60	2.85
LRu	20,20	1.45	17.35	76.80	100.	1.45	2.10	3.45	22.70
SP	20,20	1.55	16.55	72.95	100.	1.55	1.95	3.40	20.55
CLu	20,20	1.30	17.55	74.60	100.	1.30	1.85	3.00	21.55
LRu	20,15	1.20	10.50	58.10	100.	1.20	1.60	2.85	13.70
SP	20,15	1.45	10.65	57.65	100.	1.45	2.10	3.00	13.80
CLu	20,15	1.30	10.55	55.95	100.	1.30	2.00	3.30	13.45
LRu	20,10	1.05	6.10	32.20	99.40	1.05	1.35	1.90	7.50
SP	20,10	1.05	5.85	31.75	99.30	1.05	1.25	1.65	7.75
CLu	20,10	1.05	5.90	30.25	98.70	1.05	1.15	2.00	7.85

Table 5.5 continued

LRu	20,5	1.10	2.30	8.25	67.25	1.10	1.20	1.30	2.75
SP	20,5	1.05	2.35	8.10	66.85	1.05	1.05	1.30	2.55
CLu	20,5	1.00	3.10	9.80	61.50	1.00	1.25	1.55	3.55
LRu	16,16	0.85	7.25	39.65	99.95	0.85	1.10	1.55	9.00
SP	12,12	0.75	6.85	38.20	99.85	0.75	1.10	1.60	8.95
CLu	8,8	1.00	10.35	44.15	99.85	1.00	1.25	2.50	13.20
LRu	16,12	1.15	4.05	23.85	98.55	1.15	1.35	1.50	5.60
SP	12,9	1.05	4.55	24.90	98.60	1.05	1.20	1.50	5.55
CLu	8,6	1.10	7.90	32.00	98.80	1.10	1.75	2.85	[illegible]
LRu	16,8	0.80	3.40	12.30	83.00	0.80	0.90	1.10	4.15
SP	12,6	0.75	3.85	13.25	83.65	0.75	0.95	1.25	4.25
CLu	8,4	0.85	5.80	17.75	84.25	0.85	1.70	2.55	6.65
LRu	16,4	1.25	1.45	3.55	22.50	1.25	1.20	1.25	1.50
SP	12,3	1.25	1.85	4.30	26.00	1.25	1.30	1.25	2.10
CLu	8,2	1.20	3.75	9.30	36.35	1.20	1.65	2.10	4.10

Table 5.6: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.05$, $b = 0.3, 1.5$; $L = 3$.

		b = 0.3				b = 1.5			
		(u_1,u_2,u_3)				(u_1,u_2,u_3)			
Tests	n_1,r_1 n_2,n_2 n_3,r_3	(0,0,0)	(0,.1,.2)	(0,.2,.4)	(0,.5,1)	(0,0,0)	(0,.1,.2)	(0,.2,.4)	(0,.5,1)
LRu	5,5	5.15	10.25	27.20	92.15	5.15	5.35	6.35	11.65
SP	5,5	4.80	10.10	25.45	91.10	4.80	4.80	5.80	11.20
CLu	5,5	4.80	10.45	24.90	87.65	4.80	5.45	6.35	11.65
LRu	5,3	4.80	7.10	14.50	55.20	4.80	5.00	5.55	7.60
SP	5,3	4.85	6.95	14.40	54.40	4.85	4.95	5.40	7.05
CLu	5,3	5.15	7.10	15.35	54.45	5.15	4.80	5.20	7.50
LRu	10,10	5.15	17.55	57.30	100.	5.15	5.80	7.25	20.05
SP	10,10	4.90	16.85	54.70	100.	4.90	5.80	7.35	19.24
CLu	10,10	4.95	17.15	54.95	99.90	4.95	5.35	6.95	20.10
LRu	10,7	5.15	12.00	37.95	99.05	5.15	5.50	6.50	13.30
SP	10,7	4.90	12.05	37.85	98.80	4.90	5.85	6.60	14.10
CLu	10,7	5.30	12.00	36.25	97.95	5.30	5.55	6.40	14.15
LRu	10,5	4.35	9.30	25.15	91.75	4.35	4.60	5.10	10.65
SP	10,5	4.20	10.00	25.80	91.60	4.20	4.75	5.30	11.05
CLu	10,5	4.10	10.00	25.15	86.00	4.10	4.45	5.25	11.10
LRu	10,3	5.25	7.25	13.50	54.35	5.25	5.45	5.90	7.85
SP	10,3	5.20	7.50	13.90	55.40	5.20	5.10	6.00	8.05
CLu	10,3	5.65	8.25	14.95	51.50	5.65	6.25	6.10	9.00
LRu	20,20	5.50	36.70	90.25	100.	5.50	6.50	10.70	42.70
SP	20,20	5.25	33.85	88.50	100.	5.25	6.50	10.05	40.40
CLu	20,20	5.25	35.80	89.15	100.	5.25	7.00	10.45	41.35
LRu	20,15	5.50	26.50	79.05	100.	5.50	6.50	9.20	32.20
SP	20,15	5.10	25.35	78.75	100.	5.10	6.25	8.85	31.25
CLu	20,15	5.55	26.80	78.00	100.	5.55	6.15	9.05	32.30
LRu	20,10	5.25	18.25	57.25	99.90	5.25	6.00	7.30	20.70
SP	20,10	5.45	18.25	57.65	99.90	5.45	6.35	7.70	21.45
CLu	20,10	4.95	18.70	57.15	99.80	4.95	5.80	7.50	21.55

Table 5.6 continued

LRu	20,5	4.85	9.65	25.35	91.00	4.85	5.05	5.65	10.75
SP	20,5	4.80	9.85	26.20	91.15	4.80	5.15	5.90	11.00
CLu	20,5	5.20	9.95	25.90	86.55	5.20	4.75	5.80	11.35
LRu	16,16	4.65	21.10	64.05	100.	4.65	5.30	7.90	24.70
SP	12,12	4.70	20.90	62.85	100.	4.70	5.10	7.30	24.35
CLu	8,8	5.00	26.80	68.45	100.	5.00	6.90	10.55	31.00
LRu	16,12	4.65	16.25	50.30	99.85	4.65	5.40	7.00	18.30
SP	12,9	4.80	16.40	51.10	99.85	4.80	5.60	7.50	18.95
CLu	8,6	5.15	21.55	56.75	99.85	5.15	6.50	9.70	24.85
LRu	16,8	4.85	10.45	30.55	95.20	4.85	5.15	6.00	11.80
SP	12,6	4.80	11.25	32.35	95.40	4.80	5.35	6.25	12.60
CLu	8,4	5.45	15.55	40.50	95.60	5.45	6.35	8.25	17.95
LRu	16,4	4.80	6.90	13.30	52.65	4.80	4.65	5.15	7.35
SP	12,3	4.70	7.75	15.20	56.50	4.70	4.95	5.70	7.95
CLu	8,2	5.40	12.10	23.25	64.25	5.40	7.00	8.00	13.30

Table 5.7: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. α = 0.10, b = 0.3, 1.5; L = 3.

		b = 0.3				b = 1.5			
		(u_1,u_2,u_3)				(u_1,u_2,u_3)			
Tests	n_1,r_1 n_2,n_2 n_3,r_3	(0,0,0)	(0,.1,.2)	(0,.2,.4)	(0,.5,1)	(0,0,0)	(0,.1,.2)	(0,.2,.4)	(0,.5,1)
LRu	5,5	10.10	18.35	39.15	97.10	10.10	10.70	11.60	20.55
SP	5,5	10.20	18.00	38.05	96.50	10.20	11.15	12.10	19.80
CLu	5,5	10.60	17.75	37.35	95.00	10.60	10.70	11.70	19.05
LRu	5,3	10.30	13.60	24.60	71.50	10.30	10.55	10.45	14.90
SP	5,3	9.90	13.90	24.50	70.60	9.90	10.05	10.45	15.15
CLu	5,3	9.90	14.05	24.10	69.95	9.90	10.60	11.10	15.20
LRu	10,10	9.80	28.10	70.80	100.	9.80	10.20	13.30	32.30
SP	10,10	9.90	26.35	67.80	100.	9.90	10.50	13.10	30.30
CLu	10,10	9.80	28.05	70.20	100.	9.80	10.50	13.50	31.60
LRu	10,7	10.20	21.05	53.65	99.75	10.20	10.55	12.15	23.75
SP	10,7	10.20	21.60	53.55	99.60	10.20	10.50	11.95	24.15
CLu	10,7	10.40	20.55	52.95	99.40	10.40	10.25	11.60	23.75
LRu	10,5	9.35	17.45	38.95	96.50	9.35	9.45	10.25	19.15
SP	10,5	9.15	17.70	39.40	96.45	9.15	9.10	10.65	19.20
CLu	10,5	8.75	17.35	38.75	94.85	8.75	9.65	10.75	18.60
LRu	10,3	10.00	13.35	23.50	72.10	10.00	10.35	10.85	13.95
SP	10,3	10.15	13.15	23.65	72.20	10.15	10.10	10.80	14.10
CLu	10,3	10.75	13.75	23.45	66.35	10.75	10.60	10.80	14.30
LRu	20,20	9.95	47.20	94.40	100.	9.95	11.80	18.15	51.20
SP	20,20	10.35	45.00	93.55	100.	10.35	11.25	17.65	52.15
CLu	20,20	10.55	47.55	94.05	100.	10.55	12.10	18.45	53.90
LRu	20,15	10.25	39.10	87.65	100.	10.25	11.55	15.60	45.30
SP	20,15	10.05	38.25	87.30	100.	10.05	11.25	15.60	44.30
CLu	20,15	10.25	39.15	86.35	100.	10.25	11.45	15.40	45.30
LRu	20,10	10.60	28.90	70.20	100.	10.60	11.40	14.65	32.55
SP	20,10	10.45	28.50	69.75	100.	10.45	11.45	14.60	32.30
CLu	20,10	10.50	28.80	69.50	100.	10.50	11.45	14.35	32.65

Table 5.7 continued

LRu	20,5	9.85	16.65	40.20	96.60	9.85	10.10	11.40	18.05
SP	20,5	9.70	16.80	40.80	96.70	9.70	10.10	11.45	18.60
CLu	20,5	10.25	17.20	40.15	95.45	10.25	10.10	11.45	18.95
LRu	16,16	9.60	30.95	75.10	100.	9.60	10.75	14.25	35.40
SP	12,12	9.70	30.85	73.60	100.	9.70	11.15	14.50	35.55
CLu	8,8	10.25	36.45	79.25	100.	10.25	12.70	17.95	41.20
LRu	16,12	10.20	24.75	63.00	100.	10.20	11.00	13.25	28.50
SP	12,9	10.15	24.95	63.05	99.95	10.15	10.85	12.55	28.65
CLu	8,6	10.00	30.25	67.95	100.	10.00	12.20	15.65	34.85
LRu	16,8	10.70	17.95	43.50	98.40	10.70	10.95	11.85	20.25
SP	12,6	10.85	18.90	44.95	98.70	10.85	11.15	12.05	21.05
CLu	8,4	10.50	24.40	52.65	98.80	10.50	12.20	13.65	27.75
LRu	16,4	9.60	13.50	23.25	70.20	9.60	10.20	10.15	14.55
SP	12,3	9.70	14.00	24.35	72.10	9.70	9.85	10.20	15.35
CLu	8,2	9.95	18.60	33.70	78.35	9.95	11.80	13.90	20.15

Table 5.8: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.01$, $b = 0.3, 1.5$; $L = 5$.

Tests	n_1,r_1,n_2,r_2 n_3,n_3,n_4,r_4 n_5,r_5	b = 0.3 (u_1,u_2,u_3,u_4,u_5) (0,0,0,0,0)	(0,.1,.2,.3,.4)	(0.2,.4,.6,.8)	(0,.5,1,1.5,2)	b = 1.5 (u_1,u_2,u_3,u_4,u_5) (0,0,0,0,0)	(0,.1,.2,.3,.4)	(0,.2,.4,.6,.8)	(0,.5,1,1.5,2)
LRu	5,5,5,5	0.95	7.95	55.00	100.	0.95	1.00	1.45	11.10
SP	5,5,5,5	0.90	8.10	53.25	100.	0.90	0.95	1.40	10.40
CLu	5,5	0.65	8.15	44.15	99.80	0.65	0.70	1.20	10.90
LRu	5,3,5,3	1.00	3.30	17.80	92.60	1.00	1.10	1.35	3.95
SP	5,3,5,3	0.90	3.35	18.05	92.55	0.90	1.00	1.35	4.30
CLu	5,3	1.00	3.75	14.15	61.10	1.00	0.90	1.30	4.40
LRu	10,10,10,10	1.05	33.80	98.20	100.	1.05	1.65	4.60	44.15
SP	10,10,10,10	1.05	31.00	97.60	100.	1.05	1.95	4.20	40.20
CLu	10,10	0.90	30.05	95.20	100.	0.90	1.55	4.50	40.00
LRu	10,7,10,7	1.00	15.65	81.20	100.	1.00	0.95	2.55	20.15
SP	10,7,10,7	1.00	16.35	81.30	100.	1.00	1.15	2.60	21.20
CLu	10,7	0.90	14.70	71.10	100.	0.90	1.40	2.10	19.95
LRu	10,5,10,5	0.75	7.05	45.30	100.	0.75	0.65	1.50	9.20
SP	10,5,10,5	0.75	7.70	46.70	100.	0.75	0.60	1.40	10.45
10,5	0.80	8.75	39.70	99.50	0.80	0.80	1.95	10.60	
LRu	10,3,10,3	0.95	3.20	14.85	88.04	0.95	1.20	1.50	3.95
SP	10,3,10,3	1.25	3.35	15.50	87.65	1.25	1.30	1.55	4.30
CLu	10,3	1.25	8.25	49.75	57.20	1.20	1.45	1.70	4.65
LRu	20,20,20,20	1.20	80.55	100.	100.	1.20	3.00	12.35	90.10
SP	20,20,20,20	1.35	76.55	100.	100.	1.35	2.65	11.60	87.70
CLu	20,20	1.15	76.95	100.	100.	1.15	2.80	11.45	86.75
LRu	20,15,20,15	1.10	57.70	99.95	100.	1.10	1.80	6.65	70.40
SP	20,15,20,15	0.90	58.00	99.95	100.	0.90	2.10	6.95	70.35
CLu	20,15	1.30	55.30	99.90	100.	1.30	2.10	7.95	67.05
LRu	20,10,20,10	0.90	31.10	96.65	100.	0.90	1.70	3.75	41.15
SP	20,10,20,10	1.10	31.95	96.60	100.	1.10	1.60	3.75	41.20
CLu	20,10	1.10	29.00	93.60	100.	1.10	1.75	3.85	38.30

Table 5.8 continued

LRu	20,5,20,5	1.25	8.25	49.75	100.	1.25	1.40	2.00	11.30
SP	20,5,20,5	1.15	9.15	51.25	100.	1.15	1.60	2.20	11.55
CLu	20,5	1.05	8.35	40.05	99.95	1.05	1.25	1.85	10.75
LRu	16,16	0.90	41.30	99.55	100.	0.90	1.60	4.00	52.75
SP	12,12	0.75	39.75	99.60	100.	0.75	1.55	4.65	51.50
CLu	8,8	1.05	46.05	99.10	100.	1.05	2.25	6.80	58.60
LRu	16,12	1.05	25.35	94.15	100.	1.05	1.25	3.40	34.10
SP	12,9	0.95	26.80	94.35	100.	0.95	1.30	3.85	36.00
CLu	8,6	0.85	33.50	93.95	100.	0.85	2.30	6.45	41.35
LRu	16,8	0.95	12.50	67.70	100.	0.95	1.10	1.90	16.80
SP	12,6	0.85	14.15	69.65	100.	0.85	1.05	2.35	18.15
CLu	8,4	0.90	20.50	70.00	100.	0.90	2.05	3.95	24.70
LRu	16,4	0.90	3.25	17.55	92.00	0.90	0.90	1.10	3.70
SP	12,3	0.80	3.80	20.40	94.05	0.80	1.00	1.50	4.85
CLu	8,2	1.20	7.80	23.45	80.35	1.20	1.85	2.85	8.95

Table 5.9: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.05$, $b = 0.3, 1.5$; $L = 5$.

		b = 0.3				b = 1.5			
		(u_1,u_2,u_3,u_4,u_5)				(u_1,u_2,u_3,u_4,u_5)			
Tests	n_1,r_1,n_2,r_2 n_3,n_3,n_4,r_4 n_5,r_5	(0,0,0,0,0)	(0,.1,.2,.3,.4)	(0,.2,.4,.6,.8)	(0,.5,1,1.5,2)	(0,0,0,0,0)	(0,.1,.2,.3,.4)	(0,.2,.4,.6,.8)	(0,.5,1,1.5,2)
LRu	5,5,5,5	4.35	26.65	81.40	100.	4.35	4.80	7.90	32.30
SP	5,5,5,5	4.50	26.35	80.70	100.	4.50	5.00	8.30	30.70
CLu	5,5	4.30	26.50	75.25	100.	4.30	4.80	7.85	31.25
LRu	5,3,5,3	4.60	13.15	42.20	99.35	4.60	4.45	5.50	15.55
SP	5,3,5,3	4.85	13.25	42.45	99.10	4.85	4.80	6.00	15.65
CLu	5,3	5.70	13.90	36.85	93.20	5.70	5.65	6.40	15.70
LRu	10,10,10,10	4.35	58.50	99.55	100.	4.35	7.15	13.20	68.55
SP	10,10,10,10	4.80	56.05	99.50	100.	4.80	6.80	12.75	66.15
CLu	10,10	4.55	55.85	99.25	100.	4.55	7.10	13.85	65.90
LRu	10,7,10,7	4.60	38.05	95.60	100.	4.60	6.15	10.15	46.70
SP	10,7,10,7	4.60	36.45	94.85	100.	4.60	6.00	10.00	45.05
CLu	10,7	4.45	38.55	91.95	100.	4.45	5.85	10.70	45.65
LRu	10,5,10,5	4.65	23.85	77.15	100.	4.65	5.80	8.15	29.00
SP	10,5,10,5	4.50	24.30	77.20	100.	4.50	5.55	7.90	29.10
10,5	4.65	24.10	70.50	100.	4.65	5.65	8.75	28.60	
LRu	10,3,10,3	4.95	12.70	40.60	98.65	4.95	5.20	6.20	14.90
SP	10,3,10,3	5.05	13.35	41.70	98.60	5.05	5.35	6.20	15.70
CLu	10,3	5.75	14.10	38.25	92.30	5.75	5.70	6.85	16.05
LRu	20,20,20,20	5.60	92.20	100.	100.	5.60	9.65	27.65	97.35
SP	20,20,20,20	5.30	91.15	100.	100.	5.30	9.80	26.55	95.80
CLu	20,20	5.35	91.95	100.	100.	5.35	10.05	27.85	97.05
LRu	20,15,20,15	5.25	81.00	100.	100.	5.25	8.55	21.65	89.50
SP	20,15,20,15	5.50	80.75	100.	100.	5.50	8.45	21.80	88.70
CLu	20,15	4.80	78.90	100.	100.	4.80	7.90	20.45	87.25
LRu	20,10,20,10	4.40	57.45	99.50	100.	4.40	6.75	14.00	67.75
SP	20,10,20,10	4.40	57.15	99.45	100.	4.40	6.45	13.95	67.15
CLu	20,10	4.40	55.45	99.20	100.	4.40	6.45	14.05	65.10

Table 5.9 continued

LRu	20,5,20,5	4.95	23.90	78.70	100.	4.95	5.85	8.50	28.90
SP	20,5,20,5	5.15	25.25	78.95	100.	5.15	6.20	8.85	30.35
CLu	20,5	4.60	26.35	72.40	100.	4.60	5.75	8.35	31.45
LRu	16,16	4.85	68.75	99.95	100.	4.85	7.20	15.25	79.00
SP	12,12	4.85	65.85	99.95	100.	4.85	7.10	15.05	75.55
CLu	8,8	4.70	72.90	99.95	100.	4.70	8.75	20.20	81.50
LRu	16,12	4.50	52.40	98.70	100.	4.50	6.40	13.02	61.80
SP	12,9	5.00	52.60	98.80	100.	5.00	7.05	13.10	60.85
CLu	8,6	5.25	57.55	98.55	100.	5.25	8.25	16.45	65.85
LRu	16,8	5.05	33.80	89.70	100.	5.05	6.55	9.70	40.75
SP	12,6	4.85	34.55	89.80	100.	4.85	6.40	11.00	42.20
CLu	8,4	5.05	40.85	90.50	100.	5.05	7.55	13.00	46.25
LRu	16,4	4.60	13.60	42.05	99.35	4.60	5.00	5.90	15.75
SP	12,3	4.75	15.10	45.60	99.55	4.75	4.95	6.40	17.65
CLu	8,2	4.95	21.80	54.70	98.45	4.95	7.00	10.20	25.10

Table 5.10: Empirical power (%) of the statistics LRu, SP and CLu; critical values based on 10,000 replications; power based on 2000 replications. $\alpha = 0.10$, b = 0.3, 1.5; L = 5.

		b = 0.3				b = 1.5			
		(u_1,u_2,u_3,u_4,u_5)				(u_1,u_2,u_3,u_4,u_5)			
Tests	n_1,r_1,n_2,r_2 n_3,n_3,n_4,r_4 n_5,r_5	(0,0,0,0,0)	(0,.1,.2,.3,.4)	(0,.2,.4,.6,.8)	(0,.5,1,1.5,2)	(0,0,0,0,0)	(0,.1,.2,.3,.4)	(0,.2,.4,.6,.8)	(0,.5,1,1.5,2)
LRu	5,5,5,5	8.45	38.85	90.15	100.	8.45	10.30	15.20	45.05
SP	5,5,5,5	8.80	38.20	88.80	100.	8.80	10.40	14.05	44.30
CLu	5,5	9.40	37.65	86.05	100.	9.40	10.45	14.85	43.35
LRu	5,3,5,3	10.00	23.30	57.90	99.95	10.00	11.05	12.70	26.40
SP	5,3,5,3	10.20	23.55	58.55	99.95	10.20	11.25	12.95	26.65
CLu	5,3	10.20	23.60	52.45	98.45	10.20	11.15	12.80	26.70
LRu	10,10,10,10	9.70	71.15	99.80	100.	9.70	13.00	22.20	79.90
SP	10,10,10,10	10.30	68.55	99.70	100.	10.30	12.20	21.25	77.05
CLu	10,10	9.60	70.35	99.55	100.	9.60	13.05	22.15	78.45
LRu	10,7,10,7	9.55	52.05	97.90	100.	9.55	11.55	17.40	60.45
SP	10,7,10,7	9.35	51.90	97.65	100.	9.35	11.70	17.70	59.85
CLu	10,7	9.50	51.35	96.50	100.	9.50	11.75	17.50	58.85
LRu	10,5,10,5	10.20	36.65	86.80	100.	10.20	10.60	14.55	42.75
SP	10,5,10,5	10.30	37.15	86.90	100.	10.30	10.80	14.75	43.05
10,5	10.30	37.50	83.10	100.	10.30	11.20	15.85	42.95	
LRu	10,3,10,3	9.75	22.05	57.15	99.75	9.75	10.15	12.15	24.80
SP	10,3,10,3	9.45	23.20	58.30	99.85	9.45	10.70	12.25	26.20
CLu	10,3	10.25	23.70	54.35	98.15	10.25	10.50	12.35	27.40
LRu	20,20,20,20	10.35	96.80	100.	100.	10.35	18.60	40.20	98.75
SP	20,20,20,20	10.10	94.95	100.	100.	10.10	17.10	37.75	98.25
CLu	20,20	10.05	96.60	100.	100.	10.05	17.50	40.30	98.60
LRu	20,15,20,15	9.55	89.30	100.	100.	9.55	15.20	31.65	94.05
SP	20,15,20,15	9.50	88.20	100.	100.	9.50	15.35	31.75	93.65
CLu	20,15	9.45	88.45	100.	100.	9.45	15.40	33.10	93.75
LRu	20,10,20,10	9.50	69.65	99.85	100.	9.50	12.05	22.90	77.80
SP	20,10,20,10	9.30	70.00	99.85	100.	9.30	12.65	23.30	77.75
CLu	20,10	9.20	68.65	99.80	100.	9.20	12.10	23.00	77.05

Table 5.10 continued

LRu	20,5,20,5	9.35	36.50	87.05	100.	9.35	11.20	14.65	42.65
SP	20,5,20,5	9.40	36.95	87.05	100.	9.40	10.90	14.50	42.85
CLu	20,5	9.20	37.90	83.25	100.	9.20	10.75	15.65	44.10
LRu	16,16	9.90	80.35	100.	100.	9.90	13.35	26.40	88.30
SP	12,12	9.80	77.45	100.	100.	9.80	13.30	24.60	85.65
CLu	8,8	10.05	82.80	100.	100.	10.05	15.80	30.10	89.45
LRu	16,12	9.20	65.00	99.25	100.	9.20	12.60	20.90	72.60
SP	12,9	9.50	64.45	99.30	100.	9.50	12.75	21.75	73.20
CLu	8,6	9.80	69.55	99.30	100.	9.80	15.45	26.25	77.70
LRu	16,8	10.25	46.65	95.00	100.	10.25	12.30	17.90	53.30
SP	12,6	10.40	47.85	95.10	100.	10.40	12.90	18.05	54.35
CLu	8,4	9.80	53.25	95.70	100.	9.80	13.65	22.80	60.15
LRu	16,4	9.35	22.30	56.95	99.85	9.35	9.25	12.30	25.35
SP	12,3	9.25	23.60	60.10	99.90	9.25	10.05	13.00	27.05
CLu	8,2	10.15	33.45	69.20	99.75	10.15	13.00	17.10	37.35

Table 5.11: Variance of CLu in terms of $\dot{N}$ (= average sample size per group) and $\dot{R}$ (= average number of failures per group) for L = 2,...,10 groups.

L	Variance Formulae
2	$V = 0.6201 + 0.1846\,\dot{R} - 0.0100\,\dot{R}^2 + 0.0002\,\dot{R}^3 - 0.0202\,\dot{N} + 0.0006\,\dot{N}^2 - 0.000004\,\dot{N}\dot{R}$
3	$V = 1.5869 + 0.3485\,\dot{R} - 0.0190\,\dot{R}^2 + 0.0003\,\dot{R}^3 - 0.0471\,\dot{N} + 0.0010\,\dot{N}^2 + 0.0012\,\dot{N}\dot{R}$
4	$V = 2.5017 + 0.5343\,\dot{R} - 0.0313\,\dot{R}^2 + 0.0006\,\dot{R}^3 - 0.0630\,\dot{N} + 0.0013\,\dot{N}^2 + 0.0016\,\dot{N}\dot{R}$
5	$V = 3.2923 + 0.6872\,\dot{R} - 0.0336\,\dot{R}^2 + 0.0006\,\dot{R}^3 - 0.0386\,\dot{N} + 0.0012\,\dot{N}^2 - 0.0024\,\dot{N}\dot{R}$
6	$V = 4.3556 + 0.8717\,\dot{R} - 0.0405\,\dot{R}^2 + 0.0008\,\dot{R}^3 - 0.0758\,\dot{N} + 0.0033\,\dot{N}^2 - 0.0063\,\dot{N}\dot{R}$
7	$V = 6.4400 + 0.9532\,\dot{R} - 0.0433\,\dot{R}^2 + 0.0007\,\dot{R}^3 - 0.2057\,\dot{N} + 0.0072\,\dot{N}^2 - 0.0052\,\dot{N}\dot{R}$
8	$V = 6.4400 + 0.9532\,\dot{R} - 0.0433\,\dot{R}^2 - 0.0007\,\dot{R}^3 - 0.2057\,\dot{N} + 0.0072\,\dot{N}^2 - 0.0052\,\dot{N}\dot{R}$
9	$V = 9.9533 + 1.0470\,\dot{R} - 0.0385\,\dot{R}^2 + 0.0006\,\dot{R}^3 - 0.3344\,\dot{N} + 0.0114\,\dot{N}^2 - 0.0080\,\dot{N}\dot{R}$
10	$V = 11.0495 + 1.1347\,\dot{R} - 0.0478\,\dot{R}^2 + 0.0008\,\dot{R}^3 - 0.2982\,\dot{N} + 0.0092\,\dot{N}^2 - 0.0052\,\dot{N}\dot{R}$

Table 5.12: Empirical levels (%) of the statistics ACLu based on 2000 replications.

L = 2	α		
n_1,r_1,n_2,r_2	1.0	5.0	10.0
5,5,5,5	0.3	6.3	13.0
5,3,5,3	0.0	4.2	14.0
10,10,10,10	0.8	5.5	11.1
10,7,10,7	0.7	5.9	12.2
10,5,10,5	0.3	6.2	12.5
10,3,10,3	0.0	4.0	12.9
20,20,20,20	0.9	4.9	10.8
20,15,20,15	0.5	4.7	11.1
20,10,20,10	0.6	5.4	10.8
20,5,20,5	0.3	5.3	12.3
20,20,10,10	0.6	4.7	10.3
20,15,10,7	0.8	4.9	10.4
20,10,10,5	0.9	5.2	10.9
20,5,10,3	0.5	4.9	11.8

L = 3		α	
n_1,r_1,n_2,r_2,n_3,r_3	1.0	5.0	10.0
5,5,5,5,5,5	0.7	5.3	11.9
5,3,5,3,5,3	0.3	4.4	10.8
10,10,10,10,10,10	1.0	5.4	10.4
10,7,10,7,10,7	0.7	5.8	11.3
10,5,10,5,10,5	0.7	4.3	9.9
10,3,10,3,10,3	0.3	5.1	11.5
20,20,20,20,20,20	1.3	6.3	11.7
20,15,20,15,20,15	0.9	5.6	10.5
20,10,20,10,20,10	0.5	5.1	11.6
20,5,20,5,20,5	0.4	5.1	10.6
20,20,16,16,12,12	1.3	5.1	9.2
20,15,16,12,12,9	0.6	4.8	10.1
20,10,16,8,12,6	0.8	5.2	10.4
20,5,16,4,12,3	0.8	5.2	10.6

Table 5.12 continued

L = 5	α		
$n_1,r_1,n_2,r_2,n_3,r_3,n_4,r_4,n_5,r_5$	1.0	5.0	10.0
5,5,5,5,5,5,5,5,5,5	0.8	4.6	10.1
5,3,5,3,5,3,5,3,5,3	0.4	5.1	10.3
10,10,10,10,10,10,10,10,10,10	0.7	4.8	9.9
10,7,10,7,10,7,10,7,10,7	0.8	4.2	9.7
10,5,10,5,10,5,10,5,10,5	0.7	4.9	11.0
10,3,10,3,10,3,10,3,10,3	0.8	5.2	10.2
20,20,20,20,20,20,20,20,20,20	1.0	5.3	10.1
20,15,20,15,20,15,20,15,20,15	0.9	4.6	8.8
20,10,20,10,20,10,20,10,20,10	0.9	4.3	9.3
20,5,20,5,20,5,20,5,20,5	1.0	4.5	9.4
20,20,16,16,12,12,12,12,8,8	1.2	5.0	9.2
20,15,16,12,16,12,12,8,8,6	1.1	4.5	9.2
20,10,16,8,16,8,12,6,8,4	1.1	5.1	9.9
20,5,16,4,16,4,12,3,8,2	1.3	5.0	10.1

L = 10		α	
$n_1,r_1,n_2,r_2,n_3,r_3,n_4,r_4,n_5,r_5$ $n_6,r_6,n_7,r_7,n_8,r_8,n_9,r_9,n_{10},r_{10}$	1.0	5.0	10.0
5,5,5,5,5,5,5,5,5,5 5,5,5,5,5,5,5,5,5,5	0.9	5.3	10.2
5,3,5,3,5,3,5,3,5,3 5,3,5,3,5,3,5,3,5,3	0.9	4.4	9.7
10,10,10,10,10,10,10,10,10,10 10,10,10,10,10,10,10,10,10,10	0.8	4.4	9.4
10,7,10,7,10,7,10,7,10,7 10,7,10,7,10,7,10,7,10,7	0.9	4.7	9.8
10,5,10,5,10,5,10,5,10,5 10,5,10,5,10,5,10,5,10,5	0.9	4.8	10.0
10,3,10,3,10,3,10,3,10,3 10,3,10,3,10,3,10,3,10,3	1.0	4.5	9.5
20,20,20,20,20,20,20,20,20,20 20,20,20,20,20,20,20,20,20,20	1.2	5.4	10.8
20,15,20,15,20,15,20,15,20,15 20,15,20,15,20,15,20,15,20,15	0.8	5.1	9.1
20,10,20,10,20,10,20,10,20,10 20,10,20,10,20,10,20,10,20,10	0.8	4.8	9.7
20,5,20,5,20,5,20,5,20,5 20,5,20,5,20,5,20,5,20,5	0.7	4.0	8.7
20,20,16,16,12,12,12,12,8,8 20,20,16,16,16,16,12,12,8,8	1.3	5.4	10.9
20,15,16,12,16,12,12,8,8,6 20,15,16,12,16,12,12,8,8,6	1.6	6.2	10.8
20,10,16,8,16,8,12,6,8,4 20,10,16,8,16,8,12,6,8,4	1.2	6.1	11.6

Table 5.13: Empirical levels (%) and power (%) of the test statistics LRb, MB, ML, CLb and EP; Empirical levels based on 2000 replications; critical values based on 10,000 replications; power based on 2000 replications. L = 2, α = 0.01; b = 0.33.

Tests	n_1,r_1,n_2,r_2	level	Power (b_1,b_2) (.33,.33)	(.33,.66)	(.33,.99)	(.33,1.65)
LRb	5,5,5,5	3.20	1.10	5.25	16.20	45.40
MB		2.00	1.05	4.85	15.35	43.95
ML		1.65	1.15	4.85	15.35	43.95
CLb		0.00	1.20	4.90	15.65	44.25
EP			1.05	4.85	15.30	43.95
LRb	5,3,5,3	3.50	0.25	2.00	4.40	10.75
MB		0.25	0.25	2.00	4.20	10.30
ML		0.45	0.25	2.00	4.10	10.10
CLb		0.00	0.25	2.10	4.50	10.30
EP			0.25	2.00	4.20	10.75
LRb	10,10,10,10	1.70	1.35	19.85	59.60	93.80
MB		1.55	1.30	19.05	58.75	93.90
ML		1.40	1.35	19.95	59.80	93.80
CLb		1.10	1.35	19.55	59.30	92.80
EP			1.30	19.05	58.75	93.90
LRb	10,7,10,7	1.50	0.90	8.90	27.75	65.75
MB		0.85	0.85	9.35	28.05	66.40
ML		0.90	0.85	9.05	27.95	66.20
CLb		0.35	0.85	9.35	28.00	66.35
EP			0.85	9.35	28.05	66.40
LRb	10,5,10,5	3.05	1.30	5.25	14.25	37.45
MB		1.15	1.15	4.70	13.65	35.85
ML		1.45	1.20	5.15	13.95	36.80
CLb		0.00	1.20	4.95	13.40	36.35
EP			1.15	4.70	13.65	35.85
LRb	10,3,10,3	2.50	0.15	1.30	3.40	9.90
MB		0.05	0.15	1.30	3.30	9.80
ML		0.25	0.15	1.25	3.35	9.75
CLb		0.00	0.15	1.30	3.35	9.65
EP			0.15	1.30	3.30	9.80

Table 5.13 continued

LRb	20,20,20,20	1.05	0.75	53.95	94.60	100.00
MB		0.95	0.90	52.70	94.55	100.00
ML		0.80	0.75	53.80	94.60	100.00
CLb		0.75	0.75	54.45	94.80	100.00
EP			0.90	52.75	94.55	100.00
LRb	20,15,20,15	1.50	1.15	29.95	75.75	98.20
MB		1.25	1.15	29.65	76.20	98.15
ML		1.15	1.15	30.10	75.85	98.05
CLb		0.25	1.05	30.95	76.20	98.15
EP			1.15	29.65	76.20	98.15
LRb	20,10,20,10	2.05	1.10	10.05	35.15	74.85
MB		1.25	1.20	10.35	35.45	75.15
ML		1.50	1.10	10.10	35.50	75.20
CLb		0.05	1.10	10.25	35.55	75.05
EP			1.25	10.35	35.45	75.25
LRb	20,5,20,5	2.90	1.10	4.85	11.25	31.05
MB		1.00	1.10	5.10	11.40	31.45
ML		1.30	1.10	5.00	11.35	31.20
CLb		0.10	1.10	4.90	11.35	31.20
EP			1.10	5.10	11.40	31.45
LRb	20,20,10,10	1.30	0.80	31.20	77.15	98.50
MB		1.20	0.95	34.85	80.35	98.70
ML		0.85	0.85	36.95	81.10	98.70
CLb		0.90	1.15	45.25	85.90	99.05
EP			1.10	20.85	69.30	97.50
LRb	20,15,10,7	1.80	0.95	11.80	43.10	83.10
MB		0.90	0.90	17.70	51.70	87.50
ML		0.95	0.85	18.60	53.05	88.20
CLb		0.30	0.95	25.75	61.90	91.75
EP			1.10	6.45	29.45	74.45
LRb	20,10,10,5	2.70	1.20	4.60	18.15	51.25
MB		1.25	1.20	6.75	24.40	58.75
ML		1.45	1.25	7.00	25.25	59.65
CLb		0.45	1.00	13.00	36.90	59.65
EP			1.05	2.40	8.75	37.25
LRb	20,5,10,3	2.50	0.30	1.35	4.25	13.50
MB		0.20	0.30	3.00	8.50	23.05
ML		0.30	0.30	3.20	9.20	24.70
CLb		0.05	0.90	6.20	15.10	34.75
EP			0.20	0.30	1.45	5.80

Table 5.14: Empirical levels (%) and power (%) of the test statistics LRb, MB, ML, CLb and EP; Empirical levels based on 2000 replications; critical values based on 10,000 replications; power based on 2000 replications. L = 2, α = 0.05; b = 0.33.

		level	Power (b_1,b_2)			
Tests	n_1,r_1,n_2,r_2		(.33,.33)	(.33,.66)	(.33,.99)	(.33,1.65)
LRb	5,5,5,5	9.35	5.65	18.85	41.50	74.85
MB		7.15	5.65	17.95	40.95	74.65
ML		6.55	5.65	18.90	41.85	75.20
CLb		4.30	5.75	18.90	41.35	74.60
EP			5.65	17.95	40.95	74.65
LRb	5,3,5,3	11.60	4.05	8.20	16.50	31.45
MB		3.90	3.95	8.20	16.05	31.10
ML		5.25	4.05	8.15	16.45	31.45
CLb		0.10	4.00	8.25	16.80	31.10
EP			3.95	8.20	16.05	31.80
LRb	10,10,10,10	6.05	5.15	43.95	81.35	98.25
MB		5.55	5.00	44.00	81.60	98.35
ML		5.50	5.25	44.10	81.25	98.35
CLb		4.45	5.20	43.60	80.85	98.25
EP			5.00	44.00	81.65	98.35
LRb	10,7,10,7	7.50	4.80	25.05	54.05	86.55
MB		4.55	5.00	24.75	53.75	86.40
ML		5.20	4.80	25.00	53.70	86.30
CLb		3.50	4.60	24.85	53.90	86.10
EP			5.00	24.75	53.75	86.40
LRb	10,5,10,5	8.60	5.30	17.10	35.85	64.90
MB		4.85	5.30	16.55	34.85	64.25
ML		5.90	5.35	16.95	35.30	64.65
CLb		3.50	5.30	16.80	35.70	64.85
EP			5.15	16.55	34.85	64.25
LRb	10,3,10,3	10.10	3.35	7.95	15.25	29.95
MB		2.80	3.25	7.80	15.15	29.75
ML		4.30	3.35	7.90	15.25	29.95
CLb		0.05	3.35	7.85	15.35	30.05
EP			3.30	7.80	15.15	29.75

Table 5.14 continued

LRb	20,20,20,20	5.70	4.80	76.50	99.00	100.00
MB		5.25	4.60	77.00	99.05	100.00
ML		5.00	4.75	76.25	98.95	100.00
CLb		4.40	5.00	75.65	98.75	100.00
EP			4.60	77.00	99.05	100.00
LRb	20,15,20,15	6.05	5.45	54.95	90.40	99.65
MB		5.40	5.40	55.50	90.50	99.75
ML		5.65	5.40	54.80	90.35	99.65
CLb		2.20	5.20	55.05	90.55	99.65
EP			5.40	55.50	90.50	99.75
LRb	20,10,20,10	7.10	5.00	29.85	61.40	91.65
MB		5.05	5.05	29.95	61.20	91.45
ML		5.95	5.10	29.70	61.50	91.60
CLb		2.85	5.00	29.95	61.20	91.50
EP			5.05	29.95	61.60	91.45
LRb	20,5,20,5	9.20	5.50	13.90	30.95	58.35
MB		5.10	5.50	14.05	31.15	58.85
ML		5.90	5.50	13.95	31.15	58.45
CLb		3.30	5.50	14.00	31.10	58.35
EP			5.50	14.05	31.15	58.55
LRb	20,20,10,10	5.65	4.50	55.30	91.05	99.55
MB		5.05	4.50	59.10	92.30	99.70
ML		4.50	4.40	59.40	92.70	99.70
CLb		3.55	4.35	63.10	93.25	99.70
EP			4.85	50.25	88.95	99.50
LRb	20,15,10,7	6.60	4.55	28.95	65.45	93.00
MB		4.75	4.60	35.70	72.35	95.05
ML		5.25	4.70	36.00	72.30	95.10
CLb		2.70	4.95	40.35	75.25	95.95
EP			4.50	22.75	58.80	90.80
LRb	20,10,10,5	7.70	4.90	15.45	40.65	73.75
MB		5.15	5.10	21.00	47.80	79.45
ML		5.65	5.05	21.80	48.30	79.95
CLb		3.55	5.25	25.90	52.95	82.85
EP			4.70	11.00	32.80	67.00
LRb	20,5,10,3	9.30	3.45	7.10	15.30	34.75
MB		3.60	3.80	10.25	22.45	45.05
ML		4.45	3.75	10.35	22.75	45.80
CLb		1.50	3.70	15.00	30.10	52.95
EP			3.45	4.40	9.60	24.90

Table 5.15: Empirical levels (%) and power (%) of the test statistics LRb, MB, ML, CLb and EP; Empirical levels based on 2000 replications; critical values based on 10,000 replications; power based on 2000 replications. L = 2, α = 0.10; b = 0.33.

		level	Power (b_1,b_2)			
Tests	n_1,r_1,n_2,r_2		(.33,.33)	(.33,.66)	(.33,.99)	(.33,1.65)
LRb	5,5,5,5	15.05	10.45	29.65	56.45	83.80
MB		12.25	10.45	29.20	56.60	84.15
ML		12.00	10.75	29.55	56.55	83.60
CLb		9.90	10.75	30.25	56.50	84.15
EP			10.45	29.20	56.60	83.40
LRb	5,3,5,3	18.10	8.75	15.45	27.45	45.15
MB		8.75	8.75	15.55	27.15	45.15
ML		11.20	8.75	15.60	27.40	45.25
CLb		6.90	8.70	15.85	27.15	45.25
EP			8.75	15.55	27.15	45.15
LRb	10,10,10,10	11.85	9.80	58.25	89.05	99.15
MB		10.65	9.95	58.25	89.25	99.15
ML		10.25	9.90	58.40	89.15	99.15
CLb		9.10	9.80	58.30	89.50	99.15
EP			9.95	58.25	89.20	99.15
LRb	10,7,10,7	12.90	9.90	37.15	68.00	92.55
MB		9.60	10.20	37.20	67.65	92.60
ML		10.90	9.95	37.10	67.75	92.65
CLb		9.40	10.15	37.00	67.45	92.45
EP			10.20	37.20	67.65	92.60
LRb	10,5,10,5	15.45	10.55	28.70	48.90	76.50
MB		9.90	10.30	28.50	48.85	76.70
ML		11.20	10.50	28.55	49.15	76.55
CLb		9.25	10.55	28.20	48.85	76.25
EP			10.30	28.50	48.85	76.70
LRb	10,3,10,3	19.10	8.15	16.00	26.10	45.05
MB		7.20	8.10	15.95	26.15	45.00
ML		9.50	8.05	15.95	26.10	44.95
CLb		5.65	8.15	15.90	26.10	45.00
EP			8.10	15.95	26.15	45.00

Table 5.15 continued

LRb	20,20,20,20	10.50	9.95	85.65	99.40	100.00
MB		10.45	10.25	86.20	99.40	100.00
ML		9.85	9.85	85.75	99.45	100.00
CLb		9.55	9.80	85.30	99.40	100.00
EP			10.25	86.20	99.40	100.00
LRb	20,15,20,15	11.25	10.00	66.95	94.75	99.80
MB		9.75	10.05	67.30	94.85	99.80
ML		10.25	10.10	66.80	94.65	99.80
CLb		6.05	10.25	66.90	94.75	99.80
EP			10.05	67.30	94.85	99.80
LRb	20,10,20,10	12.45	10.15	42.55	74.45	95.15
MB		9.90	10.00	42.85	74.35	95.30
ML		10.80	10.10	42.55	74.40	95.25
CLb		7.80	10.15	42.85	74.45	95.20
EP			10.00	42.85	74.35	95.30
LRb	20,5,20,5	15.60	10.60	23.35	43.30	70.85
MB		10.20	10.65	23.35	43.55	71.05
ML		11.55	10.55	23.40	43.45	70.95
CLb		9.60	10.55	23.55	43.25	71.00
EP			10.60	23.30	43.50	71.05
LRb	20,20,10,10	10.95	9.50	66.55	94.40	99.75
MB		10.35	9.85	69.55	95.35	99.75
ML		10.10	9.90	70.50	95.65	99.75
CLb		8.40	9.65	71.05	95.70	99.75
EP			9.30	63.75	93.65	99.70
LRb	20,15,10,7	12.45	9.20	40.90	75.50	95.90
MB		9.20	9.30	49.95	80.55	97.00
ML		9.90	9.65	49.40	80.30	97.00
CLb		7.25	9.50	50.25	81.00	97.30
EP			8.95	35.60	72.25	95.05
LRb	20,10,10,5	14.00	8.85	26.40	52.90	82.75
MB		9.05	9.25	33.70	60.55	87.05
ML		9.65	9.30	33.85	60.70	87.05
CLb		7.45	9.25	34.50	61.20	87.30
EP			8.95	22.30	48.70	79.90
LRb	20,5,10,3	16.60	7.60	12.10	24.40	47.05
MB		7.85	8.20	19.80	34.45	57.40
ML		9.75	8.10	20.05	34.85	57.70
CLb		5.15	8.45	21.25	36.65	59.65
EP			7.25	10.00	20.00	40.85

PART II

INTERVAL ESTIMATION FOR THE PARAMETERS
OF
TWO PARAMETER
EXPONENTIAL AND EXTREME VALUE DISTRIBUTIONS
BASED ON CENSORED DATA
AND
SOME EXTENSIONS TO THE EXTREME VALUE REGRESSION MODEL

CHAPTER 6

INTERVAL ESTIMATION FOR THE PARAMETERS OF THE TWO PARAMETER EXPONENTIAL DISTRIBUTION BASED ON TIME CENSORED DATA

6.1. INTRODUCTION

In chapter 4 the main concern was with testing the homogeneity of several scale parameters of the two parameter exponential populations. In this chapter we deal with interval estimation for the scale parameter of the two parameter exponential distribution. A general discussion is given by Lawless (1982). Recall the two parameter exponential density function for time t to failure given by (2.13.3) is

$$f(t;\theta,\mu) = \frac{1}{\theta}\exp\left\{-\left(\frac{t-\mu}{\theta}\right)\right\}, \quad t \geq \mu;\ \theta > 0 \tag{6.1.1}$$

Inferences for the parameters of this model based on failure censored data have been studied by many authors (Hsieh, 1981, 1986; Singh, 1985; Lawless, 1982). In this case, exact estimation procedures for the parameters of the distribution in (6.1.1) are availabe. When the data are Type II censored, $2r\,\hat{\theta}/\theta \sim \chi^2(2r\text{-}2)$ and $2n(\hat{\mu}\text{-}\mu)/\theta \sim \chi^2(2)$ (See Lawless, 1982, P. 127), where $\hat{\mu}$ and $\hat{\theta}$ are, respectively, the MLEs of μ and θ, and r is the number of failures. Thus confidence intervals for the parameters μ and θ are easily obtained from the above results.

Under time censoring, for the threshold parameter μ, Piegorsch (1987) investigated the performance of the likelihood based interval and an interval based on the F distribution of a pivotal quantity by simulation studies. He concluded that both procedures provide confidence intervals for μ close to the nominal, with the F distribution based procedure performing slightly better for small samples. For the scale parameter θ, he examined the performance of the likelihood based interval discussed by Lawless (1982) and concluded that the upper tail probabilities are always greater than those of the lower tail; the convergence of coverage probabilities moves slowly towards the nominal levels until the sample size reaches 25.

In this chapter we develop procedures for constructing confidence intervals for θ which would provide, approximately, in small samples, equal tail probabilities and coverage probabilities close to the nominal. In chapter 4, we used the marginal likelihood to eliminate the nuisance parameter from the likelihood function for making inferences about the parameter of interest. Here, we introduce the conditional approach which is another way of eliminating the nuisance parameter μ from the likelihood, and develop procedures based on the conditional likelihood. The likelihoods to be used in section 6.3 are discussed in section 6.2. The procedures based on the conditional likelihood to construct the confidence interval for θ are derived in section 6.3. The methods are

(1) a method based on likelihood ratio.

(2) a method based on skewness corrected likelihood score (Bartlett, 1953).

(3) a method based on the mean and variance adjustments to the sign root of the likelihood ratio (Diciccio, Field and Fraser, 1990).

(4) a method based on parameter transformation to the normal approximation (Sprott, 1973).

The performance of these procedures with the usual likelihood based procedure examined by Piegrosch (1987) is studied through simulations in section 6.4. Two examples are given in section 6.5.

6.2. LIKELIHOODS

Suppose n items are placed on test and the experiment is terminated after a fixed time η_i for the ith ($i = 1,\dots,n$) item. The length of the experiment is now fixed, but the number of lifetimes observed is random. Denote the observed lifetime as t_i for the ith item. The random variable T_i, $i = 1,\dots,n$, are assumed to follow the distribution (6.1.1). Define the set of observed lifetimes $D = \{ \text{Min}\,(t_i, \eta_i);\ 1 \le i \le n \}$ and its complement by C. Denote $\delta_i = 1$ if $t_i \le \eta_i$ and $\delta_i = 0$ if $t_i > \eta_i$. The δ_i, $i = 1,\dots,n$, indicates whether the lifetime t_i is uncensored or censored. From section 2.12.2, the likelihood function under time censoring is

$$L(\mu,\theta) = \frac{1}{\theta^r} \exp\left[- \sum_{i \in D} \left(\frac{t_i - \mu}{\theta} \right) - \sum_{i \in C} \left(\frac{Max(\eta_i,\mu) - \mu}{\theta} \right) \right],$$

where $t_{(1)} = \text{Min}\,\{ t_i \}$, $i \in D$. This has been shown by Lawless (1982, p. 131). In order to restrict the number of parameters in the simulation study presented in section 6.4, we consider $\eta_i = \eta$, $i = 1,\dots,n$. Thus

$$L(\mu,\theta) = \frac{1}{\theta^r} \exp\left[- \sum_{i\in D} \left(\frac{t_i-\mu}{\theta}\right) - \sum_{i\in C} \left(\frac{Max(\eta,\mu)-\mu}{\theta} \right)\right]. \tag{6.2.1}$$

Piegorsch (1987) examined the likelihood ratio based confidence interval for θ using the likelihood $L(\mu, \theta)$ given in (6.2.1). To perform inference on θ, we eliminate the nuisance parameter μ from the likelihood $L(\mu, \theta)$ by using the conditional distribution of r and the failure times after the first, given $t_{(1)}$. If $t_{(1)} = \eta$, then $r = 0$ and no failures are observed. If $t_{(1)} < \eta$, then $P(t_{(1)} > \eta) = \exp[-n(\eta-\mu)/\theta]$ which is obtained from the marginal density of $t_{(1)}$ given as

$$f_{t_{(1)}}(t) = \frac{n}{\theta} \exp\left[- \left(\frac{t-\mu}{\theta}\right)\right] \qquad \mu < t < \eta.$$

Since $t_{(1)}$ is sufficient statistic for μ, the conditional density $f(t_{(2)},\ldots,t_{(r)} ;r \mid t_{(1)})$ is of the form

$$f(t_{(2)},\ldots,t_{(r)};\ r \mid t_{(1)}) = C_r \exp\left[- \sum_{i\in D}\left(\frac{t_{(i)}-t_{(1)}}{\theta}\right) - \sum_{i\in C} \left(\frac{\eta-t_{(1)}}{\theta}\right)\right],$$

where $C_r = \dfrac{(n-1)!}{(n-r)!\ \theta^{r-1}}$ and $t_{(1)} \le \ldots \le t_{(r)} < \eta$, which does not depend on μ. Thus

the conditional likelihood can be written as

$$L_c(\theta) = \frac{1}{\theta^{(r-1)}} \exp\left[- \sum_{i\in D} \left(\frac{t_{(i)}-t_{(1)}}{\theta}\right) - \sum_{i\in C} \left(\frac{\eta-t_{(1)}}{\theta}\right)\right] \tag{6.2.2}$$

In fact, the random variables $y_i = t_{(i+1)} - t_{(1)}$, $i = 1,...,(r-1)$, can be treated as a random sample of size (n-1) from the one parameter exponential distribution with (r-1) failures. It is clear that the procedures based on y_i are good since the likelihood $L_c(\theta)$ is independent of the unknown nuisance parameter μ.

6.3. CONFIDENCE INTERVAL PROCEDURES FOR θ

6.3.1. Likelihood Ratio Based Interval (LI)

Based on the likelihood $L(\mu,\theta)$ in (6.2.1) the maximum likelihood estimates (MLEs) $\hat{\mu}$ for μ and $\hat{\theta}$ for θ are undefined if $r = 0$. When $r > 0$, $\hat{\mu} = t_{(1)}$; $\hat{\theta} = S/r$, where

$$S = \sum_{i \in D} (t_i - \hat{\mu}) + \sum_{i \in C} Max\ \{(\eta - \hat{\mu}),\ 0\}.$$

Then the maximum log likelihood function is $l(\hat{\mu}, \hat{\theta}) = -r(\log \hat{\theta} + 1)$. Under the constraint $\theta = \theta_0$, the MLE $\tilde{\mu}$ of μ is also $t_{(1)}$. The maximum log likelihood function for the true value of θ is $l(\tilde{\mu},\theta) = -(r \log \theta + S/\theta)$. Using the estimates given above the likelihood ratio (LR) is given by

$$LR = 2\,[\, l(\hat{\mu},\hat{\theta}) - l(\tilde{\mu},\theta)] = 2r \left[\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1 \right],$$

which has approximately chi- squared distribution with one degree of freedom. The θ values that satisfy

$$2r\left\{\frac{\hat{\theta}}{\theta}-\log\left(\frac{\hat{\theta}}{\theta}\right)-1\right\}\leq\chi^2_{\alpha}(1) \tag{6.3.1}$$

are the approximate $100(1\text{-}\alpha)\%$ confidence limits for θ. Denote these limits by θ_{LL} and θ_{LU} such that $\theta_{LL} < \theta < \theta_{LU}$.

6.3.2 Conditional Likelihood Ratio Based Interval (CI)

Based on the likelihood $L_c(\theta)$ in (6.2.2) the maximum conditional likelihood estimate (MCLE) for θ is undefined if $r \leq 1$. When $r > 1$, the MCLE of θ is $\tilde{\theta} = S/(r\text{-}1)$, where S is as given in section 6.3.1. The MCLEs of μ remain $t_{(1)}$ under the constraint $\theta = \theta_0$ or not. Thus the conditional likelihood ratio statistic is

$$LR_c = 2(r-1)\left[\frac{\tilde{\theta}}{\theta}-\log\left(\frac{\tilde{\theta}}{\theta}\right)-1\right],$$

which is also approximately distributed chi- square with one degree of freedom. Thus the θ values that satisfy

$$2(r-1)\left\{\frac{\tilde{\theta}}{\theta}-\log\left(\frac{\tilde{\theta}}{\theta}\right)-1\right\}\leq\chi^2_{\alpha}(1), \tag{6.3.2}$$

are the approximate $100(1\text{-}\alpha)\%$ confidence limits for θ. Denote these limits by θ_{CL} and θ_{CU} such that $\theta_{CL} < \theta < \theta_{CU}$. This procedure is equivalent to method 2 of Lawless (1982, P 108) based on a sample of size (n-1) with (r-1) failures from a one parameter exponential distribution.

6.3.3 Skewness Corrected (Conditional Likelihood) Score Based Interval (BI)

Bartlett (1953) proposed a procedure for constructing confidence interval for a single parameter based on the likelihood score. Suppose we are interested in constructing a confidence interval for a parameter θ and let $l(\theta)$ be the log likelihood. Then $dl/d\theta$ is the likelihood score which is approximately distributed as normal with mean zero and variance $I(\theta) = E(d^2l/d\theta^2)$. Thus, an approximate confidence interval for θ can be obtained by solving

$$T_\theta = \frac{dl}{d\theta}/\sqrt{I_\theta} = \pm\,\zeta,$$

where ζ is the appropriate quantile of the standard normal distribution. Bartlett (1953) claims that this interval is asymptotically equivalent to that obtained from the MLE $\hat{\theta}$ of θ and also it has the property of providing asymptotically shortest intervals on the average. Further, he has shown that asymptotically better confidence intervals can be obtained by correcting the statistic T_θ for skewness to the order $O(n^{-1/2})$. Thus, if ζ is the $100\alpha\%$ point of the standard normal distribution then the $100(1-\alpha)\%$ approximate confidence interval for θ, using skewness corrected score, is obtained by setting

$$T_\theta - \frac{\gamma(\theta)}{6}(\zeta^2-1) = \pm\,\zeta,$$

where $\gamma(\theta)$ is the third cumulant of T_θ.

In the context of our problem, we take $l_c = l(\theta)$ as the log of the conditional likelihood (6.2.2). Using the log likelihood l_c, we obtain

$$\frac{dl_c}{d\theta} = \frac{S}{\theta^2} - \frac{(r-1)}{\theta},$$

$$\frac{d^2 l_c}{d\theta^2} = \frac{-2S}{\theta^3} + \frac{(r-1)}{\theta^2},$$

and

$$\frac{d^3 l_c}{d\theta^3} = \frac{6S}{\theta^4} - \frac{2(r-1)}{\theta^3}.$$

Under the time censoring,

$P(\delta_i = 0) = 1 - P(\delta_i = 1) = e^{-a}$, where $a = (\eta - t_{(1)})/\theta$.

For $i \in D$,

$$E\,(y_i|\ \delta_i{=}1) = \int_0^{\eta - t_{(1)}} \frac{y_i \exp(-y_i/\theta)}{\theta\ (1-e^{-\eta})} dy_i$$

$$= \frac{\theta\ (1\ -\ e^{-a}\ -\ ae^{-a})}{(1-e^{-a})}$$

and

for $i \in C$,

$E(y_i|\delta_i = 0) = (\eta - t_{(1)}) = a\theta$. Thus,

$E(y_i) = E(y_i|\delta_i{=}0)\ P(\delta_i{=}0) + E(y_i|\delta_i{=}1)\ P(\delta_i{=}1) = \theta\ (1-e^{-a})$ and

hence $E(S) = \theta\ (n-1)\ (1-e^{-a})$.

Since $E(r-1) = \sum E(\delta_i) = \sum (1-e^{-a}) = (n-1)\ (1-e^{-a})$. From these results, we obtain

$$I(\theta) = E\left(-\frac{d^2 l}{d\theta^2}\right) = \frac{(n-1)\ (1-e^{-a})}{\theta^2} = \frac{Q}{\theta^2},$$

$$\gamma(\theta) = E\left(\frac{dl}{d\theta}\right)^3 / I_\theta^{3/2} = (2Q - 3a(n-1)e^{-a}) / Q^{3/2},$$

$$where\quad Q = (n-1)\ (1-e^{-a}).$$

Thus, 100(1-α)% approximate confidence interval for θ is obtained by solving

$$\frac{\frac{S}{\theta} - (r-1)}{Q^{1/2}} - \frac{\{2Q - 3a(n-1)\ e^{-a}\}(\zeta^2-1)}{6\ Q^{3/2}} = \pm\ \zeta$$

$$or \tag{6.3.3}$$

$$\frac{S}{\theta} - (r-1) - \frac{\{2Q - 3a(n-1)\ e^{-a}\}(\zeta^2-1)}{6\ Q^{3/2}} = \pm\ \zeta\ \sqrt{Q}\ .$$

Denote the solutions by θ_{BL} and θ_{BU} such that $\theta_{BL} < \theta < \theta_{BU}$.

6.3.4 Adjusted (Conditional) Likelihood Ratio Based Interval (DI)

Diciccio, Field and Fraser (1990) developed a procedure for constructing confidence interval for a scale parameter based on adjusted likelihood ratio by using an approximate method of obtaining marginal tail probabilities of the distribution of the sign square root of the likelihood ratio statistic. Suppose $V = \log(\theta/\tilde{\theta})$ and the $A_i = (t_{(i)}-t_{(1)})/\tilde{\theta}$ for $i \in D$ and $A_i = (\eta-t_{(1)})/\tilde{\theta}$ for $i \in C$, where $\tilde{\theta}$ is as defined in section 6.3.1. Then the log conditional likelihood l_c can be written as

Since $\sum_i A_i = (r-1)$ as defined in section 6.3.2, the $l_c(V)$ reduces to $l_c(V) = -(r-1)\ (e^{-V}+V)$. It is easily seen that $l_c(V)$ is maximized at V=0, and thus the conditional likelihood ratio

$$l_c(V) = -\left\{\sum_{i=1}^{n-1} A_i\, e^{-V} + (r-1)V\right\}.$$

statistic Λ can be written as $\Lambda = 2(r-1)(e^{-V}+V-1)$. Note that $V = 0$ implies $\theta = \tilde{\theta}$. Denote

$$SR = \begin{cases} -\sqrt{\Lambda}\ , & V < 0 \\ \sqrt{\Lambda}\ , & V > 0 \end{cases}$$

$$l_1(V) = \frac{dl_c(V)}{dV} = (r-1)(e^{-V}-1),$$

and

$$l_2 = \left.\frac{d^2 l_c(V)}{dV^2}\right|_{V=0} = (r-1).$$

Following the procedure discussed in section 2.16.4, the marginal tail probability, for the pivotal V, which is given by

$$P(V \le v) = \Phi(SR) + \phi(SR)\left(\frac{1}{SR} + \frac{\sqrt{l_2}}{l_1(v)}\right) + O(n^{-3/2}), \tag{6.3.4}$$

where ϕ is the density function of a N(0,1) variable. Then the 100(1-α)% approximate lower and upper confidence limits V_L and V_U of V can be obtained by equating the expression in (6.3.4) to $\alpha/2$ and to $(1 - \alpha/2)$ respectively. It is easily shown that the 100(1-α)% approximate confidence interval for θ is given as $\tilde{\theta}\exp(V_L) < \theta < \tilde{\theta}\exp(V_U)$. Denote these limits by θ_{DL} and θ_{DU} such that $\theta_{DL} < \theta < \theta_{DU}$.

6.3.5 Interval Based on the Cube-root Transformation of $\hat{\theta}$ (SI)

Sprott (1973) and many others have indicated that the distribution of $\hat{\phi} = \hat{\theta}^{-1/3}$, where θ is the parameter of a one parameter exponential distribution, is closely approximated by a normal distribution, even for small sample sizes. In our context we use $\tilde{\phi} = \tilde{\theta}^{-1/3}$ based on the conditional likelihood (6.2.2). We can easily obtain that $E(\tilde{\phi}) = \phi$ and $var(\tilde{\phi}) = \phi^2/9Q$, where Q is as defined in section 6.3.3. Thus, the quantity $3\sqrt{\tilde{Q}}\,(\tilde{\phi} - \phi)/\tilde{\phi}$, where $\tilde{Q} = (n-1)(1- \exp(-\tilde{a}))$ and $\tilde{a} = (\eta-t_{(1)})/\tilde{\theta}$, is approximately distributed as N(0,1). Thus, the 100(1-α)% approximate confidence limits for θ is obtained by solving $3\sqrt{\tilde{Q}}\,(\tilde{\phi}-\phi)/\tilde{\phi} = \pm\,\zeta$, where ζ is the 100α% point of the standard normal distribution. It is easily seen that the 100(1-α)% approximate confidence interval for θ can be given as

$$\left\{\tilde{\phi}\left(1+\frac{\zeta}{3\sqrt{\tilde{Q}}}\right)\right\}^{-3} < \theta < \left\{\tilde{\phi}\left(1-\frac{\zeta}{3\sqrt{\tilde{Q}}}\right)\right\}^{-3}. \qquad (6.3.5)$$

Denote these limits by θ_{SL} and θ_{SU} such that $\theta_{SL} < \theta < \theta_{SU}$.

6.4 SIMULATION STUDY

In this section we study, through simulations, the performance of the confidence interval procedures presented in section 6.3. IMSL (1987) random number generator RNEXP was used to simulate two parameter exponential random variables with $\mu = 1.0$ and $\theta = 0.5$. Simulations were performed for sample sizes n = 5, 10, 25, 50 and degree of censoring π = 0.5, 0.25, 0.1, 0.0. Note that when $\pi = 0$ we deal with complete samples. The number of samples generated for each combination of n and π was taken as 1900. The censoring mechanism was the same as discussed by Piegorsch (1987), that is, the reliability at the time t, in the experiment, is given by $R(t) = \exp\{ -(t-\mu)/\theta \}$. The time η

to terminate the process was taken by setting $R(\eta) = \pi$. For all the procedures, we produce average lengths, the tail probabilities and the coverage probabilities based on the 1900 samples. The results are given in Table 6.1. Our simulations showed that for a given (n,π) combination all the results, except average lengths, of all the confidence interval procedures are invariant with respect to the choice of μ and θ. Therefore, we present our results for only one combination of (μ,θ). Note that the LI intervals are valid for $r > 0$ and CI, BI, DI and SI intervals are valid for $r > 1$. So, in simulations, for the LI intervals, samples were taken such that $r > 0$ and similarly, for the CI, BI, DI and SI intervals, samples were taken such that $r > 1$.

Results

From Table 6.1, we can see that the usual likelihood based interval LI provides the upper tail probabilities which are always larger than those of the lower tail even for sample size as large as $n = 25$ and the coverage probabilities converge slowly towards the nominal until the sample size reaches 25. This finding is similar to that of Piegorsch (1987). The performance of the intervals CI and SI is similar to that of the LI for small samples ($n = 5$). For moderate to large samples ($n = 10, 25, 50$) these intervals perform well, although they show some tendency, in some instances, for the upper tail probabilities to be larger than those of the lower tail. The interval DI shows some behaviour opposite to that of LI, namely, that the lower tail probabilities tend to be larger than those of the upper tail even for large samples ($n = 25$) and heavy censoring ($\pi = 0.5$, 0.25). For small n and large π (for example $n = 5$; $\pi = 0.25$ and $n = 10$; $\pi = 0.5$ etc.) the coverage probabilities fall short of the nominal coverage. The performance of the skewness corrected score interval BI seems best in that it provides reasonably accurate lower and upper tail probabilities and coverage probabilities close to the nominal even for sample sizes as small as n = 5, except for n = 5 and $\pi = 0.5$. For n = 5 and $\pi = 0.5$ all procedures perform poorly. For the LI interval, the results in Table 2 and 3 of Piegorsch (1987) show similar performance. The reason for the poor performance is that although samples are obtained with average 50% censoring the actual percentage censoring, because of the conditions $r > 0$ for the LI and $r > 1$ for the BI, CI, DI and SI, is more than 50%. The effect of this diminishes as the sample size increases or the percentage censoring decreases. Considering the lengths of the confidence intervals we find that the mean of the average lengths is the smallest for the LI interval and largest for the DI interval. The mean of the average lengths for the intervals CI, BI and SI are almost indistinguishable (in the computation of the mean of average lengths we did not consider n = 5 and $\pi = 0.5$). Thus, in terms of both holding nominal coverage probability and tail symmetry, the interval BI seems to perform best. The usual likelihood interval LI maintains, on the average, the shortest length. But it has the disadvantage that unless the sample size is large it does not maintain the nominal coverage and yields asymmetric tail probabilities.

6.5. EXAMPLES

Example 1: Confidence limits obtained by all the methods discussed, are given here for the transistor data (n = 34 transistors) considered by Piegrosch (1987). With a fixed censoring time of $\eta = 40$ weeks, the observed lifetimes to the nearest weeks are 3, 4, 5, 6, 7, 8, 8, 9, 9, 10, 10, 11, 11, 11, 13, 13, 13, 13, 13, 17, 17, 19, 19, 25, 29 and 33. Thus, r = 28 and under a two parameter exponential model, we obtain $\hat{\mu}$ =3; $\hat{\theta}$ =17.464; $\tilde{\theta}$ = 18.111. Choosing α = 0.05 implies $\chi^2(1)$ =3.841 and ζ =1.96. For the 95% confidence intervals for θ we obtain $\hat{\theta}_{LL}$ = 12.321, $\hat{\theta}_{LU}$ = 25.917, $\hat{\theta}_{CL}$ =12.700, $\hat{\theta}_{CU}$ = 27.805, $\hat{\theta}_{BL}$ = 12.647, $\hat{\theta}_{BU}$ = 27.771, $\hat{\theta}_{DL}$ = 12.830, $\hat{\theta}_{DU}$ = 27.491, $\hat{\theta}_{SL}$ = 12.367 and $\hat{\theta}_{SU}$ = 25.795. The lengths of the LI, CI, BI, DI and SI intervals are respectively 13.595, 14.385, 15.124, 14.661 and 13.428.

Example 2: The data set by Bartholomew (1963) is considered here. With a fixed censoring time of η = 150 hours, 20 items were placed on lifetest and 15 items failed with the observed lifetimes, in hours, 3, 19, 23, 27, 37, 38, 41, 45, 58, 84, 90, 109 and 138. Under the two parameter exponential model, we obtain $\hat{\mu}$ = 3; $\hat{\theta}$ = 101.800 and $\tilde{\theta}$ = 109.071. Choosing α = 0.05 employs $\chi^2(1)$ =3.841 and ζ = 1.96. For the 95% confidence interval for θ we obtain $\hat{\theta}_{LL}$ = 63.834, $\hat{\theta}_{LU}$ = 176.897, $\hat{\theta}_{CL}$ = 67.371, $\hat{\theta}_{CU}$ = 193.601, $\hat{\theta}_{BL}$ = 67.582, $\hat{\theta}_{BU}$ = 196.068, $\hat{\theta}_{DL}$ = 68.681, $\hat{\theta}_{DU}$ = 199.520, $\hat{\theta}_{SL}$ = 62.879 and $\hat{\theta}_{SU}$ = 180.780. The lengths of the LI,CI, BI, DI and SI intervals are respectively 113.063, 126.230, 128.486, 130.839 and 117.901.

Table 6.1: Average lengths, lower and upper tail probabilities
and coverage probabilities of the confidence intervals
LI, *CI*, *BI*, *DI* and *SI*. $\mu = 1.0$, $\theta = 0.5$.

n	π	The Procedures	$\alpha = 0.10$ Length	lower	upper	coverage	$\alpha = 0.05$ Length	lower	upper	coverage
5	.5	*LI*	1.953	0.000	0.180	0.820	7.150	0.000	0.105	0.895
		CI	3.956	0.000	0.063	0.937	7.150	0.000	0.033	0.967
		BI	3.524	0.000	0.055	0.945	7.655	0.000	0.033	0.967
		DI	7.950	0.000	0.042	0.958	15.867	0.000	0.025	0.975
		SI	4.518	0.000	0.062	0.938	9.723	0.000	0.030	0.970
	.25	*LI*	1.231	0.005	0.156	0.839	1.791	0.003	0.105	0.893
		CI	2.995	0.056	0.075	0.869	4.992	0.015	0.041	0.944
		BI	2.817	0.046	0.058	0.896	5.379	0.010	0.018	0.972
		DI	5.219	0.077	0.055	0.867	9.659	0.034	0.028	0.938
		SI	3.258	0.058	0.068	0.873	6.158	0.013	0.038	0.949
	.1	*LI*	0.894	0.013	0.167	0.820	1.193	0.008	0.107	0.884
		CI	1.724	0.050	0.076	0.874	2.549	0.021	0.040	0.939
		BI	1.764	0.049	0.040	0.910	2.827	0.020	0.011	0.969
		DI	2.416	0.068	0.056	0.876	3.862	0.039	0.028	0.933
		SI	1.800	0.049	0.074	0.877	2.822	0.020	0.039	0.941
	0	*LI*	0.716	0.010	0.167	0.823	0.928	0.004	0.107	0.889
		CI	1.052	0.035	0.075	0.890	1.394	0.015	0.040	0.945
		BI	1.308	0.042	0.047	0.912	1.720	0.023	0.024	0.954
		DI	1.204	0.046	0.056	0.898	1.607	0.024	0.028	0.948
		SI	1.065	0.035	0.075	0.890	1.422	0.015	0.040	0.945
10	.5	*LI*	1.143	0.015	0.112	0.873	1.631	0.005	0.064	0.931
		CI	2.289	0.052	0.052	0.896	3.642	0.026	0.027	0.947
		BI	2.089	0.044	0.052	0.903	3.671	0.020	0.025	0.955
		DI	3.693	0.086	0.042	0.873	3.975	0.038	0.020	0.942
		SI	2.452	0.048	0.051	0.901	4.354	0.024	0.024	0.952
	.25	*LI*	0.693	0.022	0.103	0.875	0.884	0.010	0.058	0.932
		CI	0.957	0.054	0.052	0.894	1.217	0.024	0.027	0.949
		BI	0.954	0.051	0.048	0.901	1.242	0.024	0.019	0.957
		DI	1.102	0.073	0.042	0.885	1.494	0.037	0.020	0.943
		SI	0.975	0.057	0.050	0.892	1.246	0.023	0.026	0.951
	.1	*LI*	0.572	0.020	0.110	0.870	0.715	0.010	0.063	0.927
		CI	0.700	0.046	0.058	0.895	0.882	0.021	0.029	0.950
		BI	0.722	0.050	0.042	0.908	0.923	0.022	0.020	0.958
		DI	0.749	0.058	0.044	0.897	0.946	0.029	0.023	0.948
		SI	0.709	0.045	0.056	0.898	0.896	0.020	0.028	0.952
	0	*LI*	0.516	0.020	0.111	0.869	0.641	0.009	0.065	0.926
		CI	0.611	0.038	0.059	0.903	0.762	0.019	0.029	0.952
		BI	0.626	0.044	0.042	0.914	0.821	0.023	0.022	0.955
		DI	0.646	0.045	0.045	0.909	0.807	0.024	0.023	0.953
		SI	0.613	0.038	0.059	0.903	0.767	0.018	0.029	0.953

Table 6.1: Continued

n	π	The Procedures	$\alpha = 0.10$ Length	lower	upper	coverage	$\alpha = 0.05$ Length	lower	upper	coverage
25	.5	*LI*	0.519	0.023	0.074	0.903	0.643	0.014	0.038	0.948
		CI	0.609	0.048	0.042	0.910	0.758	0.023	0.022	0.955
		BI	0.603	0.046	0.042	0.912	0.755	0.023	0.021	0.956
		DI	0.642	0.065	0.035	0.900	0.801	0.032	0.017	0.951
		SI	0.611	0.046	0.040	0.914	0.763	0.024	0.022	0.954
	.25	*LI*	0.395	0.027	0.078	0.895	0.481	0.013	0.038	0.949
		CI	0.432	0.043	0.046	0.912	0.527	0.023	0.019	0.958
		BI	0.434	0.046	0.042	0.913	0.533	0.023	0.017	0.960
		DI	0.444	0.053	0.038	0.908	0.543	0.028	0.017	0.955
		SI	0.433	0.042	0.044	0.914	0.529	0.022	0.020	0.958
	.1	*LI*	0.351	0.025	0.083	0.892	0.425	0.009	0.044	0.947
		CI	0.376	0.040	0.055	0.905	0.457	0.018	0.026	0.956
		BI	0.382	0.048	0.047	0.905	0.466	0.019	0.022	0.959
		DI	0.385	0.047	0.046	0.907	0.468	0.023	0.021	0.956
		SI	0.378	0.040	0.054	0.906	0.459	0.018	0.025	0.957
	0	*LI*	0.327	0.022	0.083	0.896	0.396	0.007	0.044	0.949
		CI	0.348	0.040	0.055	0.905	0.422	0.015	0.025	0.960
		BI	0.357	0.047	0.045	0.908	0.433	0.018	0.022	0.960
		DI	0.355	0.047	0.047	0.906	0.431	0.018	0.023	0.959
		SI	0.349	0.040	0.055	0.905	0.423	0.015	0.025	0.960
50	.5	*LI*	0.343	0.028	0.078	0.894	0.416	0.010	0.038	0.952
		CI	0.367	0.046	0.051	0.903	0.445	0.022	0.027	0.951
		BI	0.366	0.046	0.048	0.906	0.445	0.022	0.027	0.951
		DI	0.375	0.056	0.046	0.898	0.455	0.027	0.025	0.948
		SI	0.368	0.045	0.051	0.904	0.446	0.021	0.027	0.952
	.25	*LI*	0.272	0.027	0.078	0.895	0.328	0.015	0.039	0.946
		CI	0.284	0.041	0.057	0.902	0.342	0.018	0.027	0.955
		BI	0.285	0.043	0.055	0.902	0.344	0.018	0.025	0.957
		DI	0.288	0.048	0.052	0.900	0.347	0.020	0.024	0.956
		SI	0.284	0.041	0.056	0.903	0.343	0.019	0.025	0.956
	.1	*LI*	0.246	0.034	0.079	0.887	0.296	0.016	0.037	0.947
		CI	0.255	0.045	0.054	0.901	0.306	0.022	0.026	0.952
		BI	0.256	0.047	0.050	0.903	0.309	0.024	0.024	0.952
		DI	0.257	0.050	0.050	0.900	0.301	0.027	0.023	0.950
		SI	0.255	0.046	0.053	0.901	0.307	0.023	0.024	0.952
	0	*LI*	0.231	0.028	0.075	0.897	0.278	0.013	0.036	0.951
		CI	0.239	0.040	0.058	0.902	0.287	0.017	0.026	0.957
		BI	0.242	0.044	0.050	0.906	0.290	0.021	0.024	0.955
		DI	0.241	0.044	0.051	0.905	0.289	0.021	0.024	0.955
		SI	0.239	0.040	0.058	0.902	0.287	0.017	0.026	0.957

CHAPTER 7

CONFIDENCE INTERVAL ESTIMATION FOR THE PARAMETERS OF EXTREME VALUE MODELS UNDER FAILURE CENSORING

7.1 INTRODUCTION

The previous chapter describes several interval estimation procedures for the scale parameter of the two parameter exponential distribution under time censoring. This chapter concerns setting approximate confidence intervals for the parameters of the two parameter extreme value distribution as well as the extreme value regression model with failure censored data. A number of procedures exist for the construction of confidence interval for the location- scale parameters of an extreme value distribution having pdf (2.13.5) for failure censored data (Mann, Fertig and Schefer, 1971; Mann and Fertig, 1979; McCool, 1970, 1974; Lawless, 1978, 1982; Mann, Schefer and Singpurwalla, 1974). Mann and Fertig (1977) propose confidence intervals based on linear invariant estimators. The linear invariant estimators rely on tables and the critical values for the relevant pivotal quantities need to be evaluated by simulations. However, distributions of the pivotals depend only on n and r (n is the sample size; r is the number of failures) and they tabulate percentage points for $3 \leq r \leq n \leq 25$. McCool (1970, 1974) gives similar types of tables for pivotals employing maximum likelihood estimators (MLEs). All these methods are exact but they require extensive tables. Lawless (1978, 1982) proposes a conditional approach which is also exact, but it requires extensive computations involving numerical integrations. For complete samples Bain and Engelhardt (1981) propose an approximate method which is reasonably simple to use. Peers and Iqbal (1985) develop a procedure based on Fisher-Cornish expansions in a similar situation. Relevant previous work on regression analysis based on the extreme value distribution with failure censored data includes Nelson and Hahn (1972, 1973); Lawless (1976, 1982) and McCool (1980). Nelson and Hahn (1972, 1973) provide best linear unbiased estimates and simple linear unbiased estimates and approximate confidence limits for the parameters of extreme value regression model with failure censored data. Lawless (1976) considers the power law model and uses the conditional approach to provide the confidence interval estimates for the power law model parameters. Lawless (1982) presents a theory for exact confidence limits for the parameters of the extreme value regression model, which require extensive computations involving numerical integrations. For a similar type of problem, McCool (1980) develops procedures based on appropriate pivotals whose distributions depend upon the sample sizes and the number of failures. These procedures are applicable only to equal sample sizes and to equal number of lifetimes. Recently, Diciccio, Field and Fraser (1990) propose an approximate method based on mean and variance corrected signed root of the likelihood ratio statistic in constructing confidence intervals for the parameters of location- scale models.

In section 7.2, the maximum likelihood estimators of the parameters for a single sample from the extreme value model are reviewed. In section 7.3, the exact and approximate expected Fisher information matrices for the MLEs of the parameters of both the two parameter extreme value distribution and the extreme value regression model with failure censored data are obtained and compared. The elements of approximate expected

Fisher information matrix have mathematically and computationally simple expressions. Cox and Reid (1987) give a general procedure for the construction of orthogonal parameters. By using this approach we obtain explicit expressions for the asymptotic variance-covariance of the MLEs of the parameters in section 7.4. For the parameters of interest various interval estimation procedures are developed in section 7.5. These include a method based on asymptotic properties of the MLEs, a method based on the likelihood ratio statistic, a method based on mean and variance adjustments to the signed root of the likelihood ratio statistic and a method based on the likelihood score corrected for bias and skewness. We then compare them in terms of average lengths, tail probabilities and coverage probabilities by simulations. These procedures are extended to the extreme value regression model in section 7.6.

7.2 MAXIMUM LIKELIHOOD ESTIMATION

7.2.1 Two Parameter Extreme Value Distribution

Consider a sample of size n from an extreme value distribution with pdf (2.13.5). Let Y_j denote the jth ordered observation. Suppose the first r smallest observations are observed. The log likelihood, apart from an irrelevant constant, is

$$l(u,b) = -r \log b + \sum_{j=1}^{r} \left(\frac{Y_j - u}{b} \right) - \sum_{j=1}^{*} \exp\left(\frac{Y_j - u}{b} \right), \tag{7.2.1}$$

where

$$\sum_{j=1}^{*} W_j = \sum_{j=1}^{r} W_j + (n-r)\, W_r.$$

As discussed in section 2.2, the maximum likelihood equations are given by

$$\frac{\partial l}{\partial u} = \frac{1}{b}\left\{ \sum_{j=1}^{*} \exp\left(\frac{Y_i-u}{b} \right) - r \right\} = 0 \tag{7.2.2}$$

and

$$\frac{\partial l}{\partial b} = \frac{1}{b}\left\{ \sum_{j=1}^{*} \left(\frac{Y_j-u}{b} \right) \exp\left(\frac{Y_j-u}{b} \right) - \sum_{j=1}^{r} \left(\frac{Y_j-u}{b} \right) - r \right\} = 0. \tag{7.2.3}$$

From (7.2.2), we obtain

$$u = b \log\left[\frac{1}{r} \sum_{j=1}^{*} \exp\left(\frac{Y_j}{b}\right) \right]. \tag{7.2.4}$$

Substituting the value of u in equation (7.2.3) yields

$$\frac{\sum_{j=1}^{*} Y_j \exp\left(\frac{Y_j}{\hat{b}}\right)}{\sum_{j=1}^{*} \exp\left(\frac{Y_j}{\hat{b}}\right)} - \sum_{j=1}^{r} \frac{Y_j}{r} - \hat{b} = 0\ . \tag{7.2.5}$$

The MLE of the scale parameter b, denoted by $\hat{b}$, is computed by solving equation (7.2.5) iteratively. Having obtained $\hat{b}$, the MLE of u is calculated from (7.2.4) which is given by

$$\hat{u} = \hat{b} \log \left[\frac{1}{r} \sum_{j=1}^{*} \exp \left(\frac{Y_j}{\hat{b}}\right) \right]. \tag{7.2.6}$$

7.2.2 Extreme Value Regression Model

We consider the regression model in which the location parameter u is a linear combination of the regressor variables $X_1,...,X_m$ such that $u(X) = \beta_1 X_1 + ... + \beta_m X_m$ with $X_1 = 1$, where $\beta = (\beta_1,...,\beta_m)'$ is the vector of m regression coefficients to be estimated from the available data. The log likelihood (7.2.1) can now be written as

$$l(\beta,b) = -r \log b + \sum_{j=1}^{r} \left(\frac{Y_j - X_j\beta}{b} \right) - \sum_{j=1}^{*} \exp \left(\frac{Y_j - X_j\beta}{b} \right) \tag{7.2.7}$$

where $X_j\beta = \beta_1 X_{j1} + \beta_2 X_{j2} + ... + \beta_m X_{jm}$. The maximum likelihood equations are

$$\frac{\partial l}{\partial \beta_p} = \frac{1}{b} \left[\sum_{j=1}^{*} X_{jp} \exp \left(\frac{Y_j - X_j\beta}{b} \right) - \sum_{j=1}^{r} X_{jp} \right] = 0, \tag{7.2.8}$$

for p = 1,...,m, and

$$\frac{\partial l}{\partial b} = \frac{1}{b} \left[\sum_{j=1}^{*} \left(\frac{Y_j - X_j\beta}{b} \right) \exp \left(\frac{Y_j - X_j\beta}{b} \right) - \sum_{j=1}^{r} \left(\frac{Y_j - X_j\beta}{b} \right) - r \right] = 0\ . \tag{7.2.9}$$

Solving the above (m+1) equations simultaneously yields the MLE of $\beta_1,...,\beta_m$ and b. Denote these estimates by $\beta_1,...,\beta_m$ and $\hat{b}$ respectively.

7.3 FISHER INFORMATION MATRIX

7.3.1 Two Parameter Extreme Value Distribution

From the log likelihood (7.2.1), the negative mixed partial derivatives are obtained as follows:

$$-\frac{\partial^2 l}{\partial u^2} = \frac{1}{b^2}\sum_{j=1}^{*} \exp\left(\frac{Y_j-u}{b}\right),$$

$$-\frac{\partial^2 l}{\partial u \partial b} = \frac{1}{b^2}\left\{\sum_{j=1}^{*}\left(\frac{Y_j-u}{b}+1\right)\exp\left(\frac{Y_j-u}{b}\right)-r\right\}$$

and

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{b^2}\left\{\sum_{j=1}^{*}\left[\left(\frac{Y_j-u}{b}\right)^2+2\left(\frac{Y_j-u}{b}\right)\right]\exp\left(\frac{Y_j-u}{b}\right)-2\sum_{j=1}^{r}\left(\frac{Y_j-u}{b}\right)-r\right\}.$$

Let $Z_j = (Y_j-u)/b$, $j = 1,\ldots,r$, then using the exact expression for the terms $E(Z_j)$, $E(\exp(Z_j))$, $E(Z_j \exp(Z_j))$ and $E(Z_j^2 \exp(Z_j))$ given in section 2.14, we obtain

$$b^2 E\left(-\frac{\partial^2 l}{\partial u^2}\right) = A = \sum_{j=1}^{*} c_j \left\{\sum_{s=1}^{j} (-1)^{s-1}\binom{j-1}{s-1}\frac{1}{(n-j+s)^2}\right\} - r,$$

$$b^2 E\left(-\frac{\partial^2 l}{\partial u \partial b}\right) = C = \sum_{j=1}^{*} c_j \left\{\sum_{s=1}^{j} (-1)^{s-1}\binom{j-1}{s-1}\frac{(2-\gamma-\log(n-j+s))}{(n-j+s)^2}\right\} - r$$

and

$$b^2\, E\left(-\frac{\partial^2 l}{\partial b^2} \right) = J$$

$$= \sum_{j=1}^{*} c_j \left\{ \sum_{s=1}^{j} (-1)^{s-1} \binom{j-1}{s-1} \left[\frac{(\frac{\pi^2}{6}-2) + (2-\gamma-\log(n-j+s))^2}{(n-j+s)^2} \right] \right\}$$

$$- 2 \sum_{j=1}^{r} c_j \left\{ \sum_{s=1}^{j} (-1)^{s-1} \binom{j-1}{s-1} \left(\frac{\gamma+\log(n-j+s)}{(n-j+s)} \right) \right\} - r,$$

(7.3.2)

where $\gamma = 0.5772...$ is the Euler's constant. It is clear that the exact expected values of these mixed partial derivatives are mathematically and computationally messy. Approximate but very accurate expected values can be obtained by using the approximate expressions of $E(Z_j)$, $E(\exp(Z_j))$, $E(Z_j \exp(Z_j))$ and $E(Z^2_j \exp(Z_j))$ given in section 2.14. The approximate expected values are

$$b^2\, E\left(-\frac{\partial^2 l}{\partial u^2} \right) = \sum_{j=1}^{*} t_j = r,$$

$$b^2\, E\left(-\frac{\partial^2 l}{\partial u \partial b} \right) \approx \sum_{j=1}^{*} \left(t_j \log t_j + \frac{d_j}{2t_j} \right)$$

and

$$b^2\, E\left(-\frac{\partial^2 l}{\partial b^2} \right) = \sum_{j=1}^{*} (2 + \log t_j)(t_j \log t_j + \frac{d_j}{t_j})$$

$$- \sum_{j=1}^{r} (2 \log t_j + \frac{d_j}{t_j^2}) - r\,.$$

It is easily seen from (7.3.2) and (7.3.3) that A = r,

$$C = \sum_{j=1}^{*} (t_j \log t_j + \frac{d_j}{2t_j})$$

and (7.3.3)

$$J = \sum^{*} (2 + \log t_j)(t_j \log t_j + \frac{d_j}{t_j}) - \sum_{j=1}^{r} (2 \log t_j - \frac{d_j}{t_j^2}) - r.$$

For complete sample, we replace r by n and obtain

$$b^2\, E\left(-\frac{\partial^2 l}{\partial u^2} \right) = \sum_{j=1}^{n} E\left[\exp\left(\frac{y_j - u}{b} \right) \right] = n,$$

$$b^2\, E\left(-\frac{\partial^2 l}{\partial u \partial b} \right) = \sum_{j=1}^{n} E\left\{ \left[\left(\frac{y_j - u}{b} \right) + 1 \right] \exp\left(\frac{y_j - u}{b} \right) - n \right\} = \frac{n\,(1-\gamma)}{b^2}$$

and

$$b^2 E\left(-\frac{\partial^2 l}{\partial b^2}\right) = \sum_{j=1}^{n} E\left\{\begin{array}{c}\left[\left(\frac{y_j-u}{b}\right)^2 + 2\left(\frac{y_j-u}{b}\right)\right]\exp\left(\frac{y_j-u}{b}\right) \\ -2\left(\frac{y_j-u}{b}\right) - n\end{array}\right\} \tag{7.3.4}$$

$$= n\ K,$$

where $K = \pi^2/6 + (1-\gamma)^2$. The exact and approximate values for C and J obtained from (7.3.2) and (7.3.3) respectively, are given for various selected combinations of (n,r) in Table 7.1.

Table 7.1

Exact and Approximate values of C and J for a single sample; n = sample size; r = number of failures.

n	r	Exact C	Exact J	Approximate C	Approximate J
5	5	2.114	9.125	2.184	9.179
5	3	-0.689	4.447	-0.632	4.382
10	10	4.228	18.250	4.283	18.221
10	7	-0.630	9.655	-0.596	9.505
10	5	-2.385	7.721	-2.352	7.564
10	3	-2.973	6.979	-2.936	6.830
20	20	8.456	36.500	8.497	36.396
20	15	-0.289	20.329	-0.268	20.113
20	10	-5.101	15.068	-5.082	14.849
20	5	-6.177	13.829	-6.153	13.624
30	15	-7.824	22.394	-7.810	22.151

As can be seen, the exact and approximate values are similar. As defined in section 2.3, the Fisher information matrix is given by

$$I = \begin{bmatrix} E\left(-\dfrac{\partial^2 l}{\partial u^2} \right) & E\left(-\dfrac{\partial^2 l}{\partial u \partial b} \right) \\ E\left(-\dfrac{\partial^2 l}{\partial b \partial u} \right) & E\left(-\dfrac{\partial^2 l}{\partial b^2} \right) \end{bmatrix} = \frac{1}{b^2} \begin{bmatrix} A & C \\ C & J \end{bmatrix} . \qquad (7.3.5)$$

7.3.2 Extreme Value Regression Model

From the log likelihood (7.2.6), we obtain

$$-\frac{\partial^2 l}{\partial \beta_p \partial \beta_q} = \frac{1}{b^2} \sum_{j=1}^{*} X_{jp} X_{jq} \exp\left(\frac{Y_j - X_j \beta}{b}\right)$$

$$-\frac{\partial^2 l}{\partial \beta_p \partial b} = \frac{1}{b^2} \left\{ \sum_{j=1}^{*} X_{jp} \left(\frac{Y_j - X_j \beta}{b} + 1 \right) \exp\left(\frac{Y_j - X_j \beta}{b}\right) - \sum_{j=1}^{r} X_{jp} \right\}$$

and

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{b^2} \left\{ \sum_{j=1}^{*} \left[\left(\frac{Y_j - X_j \beta}{b}\right)^2 + 2\left(\frac{Y_j - X_j \beta}{b}\right) \right] \exp\left(\frac{Y_j - X_j \beta}{b}\right) \right\}$$

$$- \frac{2}{b^2} \sum_{j=1}^{r} \left(\frac{Y_j - X_j \beta}{b}\right) - \frac{r}{b^2} .$$

Since $Z_j = (Y_j - u)/b = (Y_j - X_j\beta)/b$, using the approximate expressions for $E(Z_j)$, $E(\exp(Z_j))$, $E(Z_j \exp(Z_j))$ and $E(Z_j^2 \exp(Z_j))$ given in section 2.14, we obtain, for p, q = 1,...,m,

$$A_{pq} = b^2\, E\left(- \frac{\partial^2 l}{\partial\beta_p \partial\beta_q} \right) = \sum^{*} X_{jp}\, X_{jq}\, t_j,$$

$$C_p = b^2\, E\left(- \frac{\partial^2 l}{\partial\beta_p \partial b} \right) = \sum^{*} X_{jp} \left(t_j\, (1+\log t_j) + \frac{d_j}{2t_j} \right) - \sum_{j=1}^{r} X_{jp}$$

and

$$J = b^2\, E\left(- \frac{\partial^2 l}{\partial b^2} \right)$$

$$= \sum_{j=1}^{*} (2+\log t_j)\, (t_j \log t_j + \frac{d_j}{t_j}\,) - \sum_{j=1}^{r} (2 \log t_j - \frac{d_j}{t_j^2}) - r.$$

Then the Fisher information matrix, approximately, is given by

$$I = \frac{1}{b^2} \begin{bmatrix} A_{11} & . & . & . & A_{1m} & C_1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{m1} & . & . & . & A_{mm} & C_m \\ C_1 & . & . & . & C_m & J \end{bmatrix}. \qquad (7.3.7)$$

For the complete sample, for p,q = 1,...,m, and j = 1,...,n,

$A_{pq} = \sum X_{jp}\, X_{jq}$,

$C_p = \sum X_{jp}\, (1-\gamma)$, and J = nK,

where K is as defined earlier. These results concur with the results of Lawless (1982, P. 301).

7.4 ASYMPTOTIC VARIANCE-COVARIANCE OF THE MLEs

7.4.1 Two Parameter Extreme Value Distribution

Following section 2.4, the asymptotic variance-covariance of the MLEs of the extreme value distribution parameters can be obtained by inverting the Fisher information matrix as defined in (7.3.5). In this section we employ the orthogonal approach discussed by Cox and Reid (1987) to obtain explicit expressions for the variance-covariances of the MLEs of the parameters u and b.

Orthogonality is defined with respect to the expected (Fisher) information matrix. A parameter λ orthogonal to the scale parameter b is obtained by solving the partial differential equation (equation (4) of Cox and Reid, 1987)

$$E\left(- \frac{\partial^2 l}{\partial u^2} \right) \frac{\partial u}{\partial b} = - E\left(- \frac{\partial^2 l}{\partial u \partial b} \right),$$

where $u = u(\lambda, b)$; that is

$$A\ \partial u / \partial b = - C, \tag{7.4.1}$$

where A and C are as given in section 7.3 which do not depend on the parameters u and b. Thus, the solution of the differential equation (7.4.1) is easily given by

$$\lambda = Au + Cb. \tag{7.4.2}$$

Now, we express the log likelihood function (7.2.1) in terms of the parameters λ and b. Since $A = r$, as shown in section 7.3, we have

$$l(\lambda,b) = -r\log b + \sum_{j=1}^{r}\left(\frac{rY_j-\lambda+Cb}{rb}\right) - \sum_{j=1}^{*} \exp\left(\frac{rY_j-\lambda+Cb}{rb}\right). \tag{7.4.3}$$

From (7.4.3), we obtain

$$-\frac{\partial^2 l}{\partial\lambda^2} = \frac{1}{r^2\, b^2}\sum_{j=1}^{*} \exp\left(\frac{rY_j\ -\lambda+Cb}{rb}\right),$$

$$-\frac{\partial^2 l}{\partial\lambda\partial b} = \frac{1}{r\, b^2}\sum_{j=1}^{*}\left(\frac{rY_j\ -\lambda}{rb}+1\right)\exp\left(\frac{rY_j\ -\lambda+Cb}{rb}\right) - \frac{1}{b^2}$$

and

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{b^2}\sum_{j=1}^{*}\left[\left(\frac{rY_j-\lambda}{rb}\right)^2 + 2\left(\frac{rY_j-\lambda}{rb}\right)\right]\exp\left(\frac{rY_j-\lambda+Cb}{rb}\right) \tag{7.4.4}$$
$$-\frac{2}{b^2}\sum_{j=1}^{r}\left(\frac{rY_j-\lambda}{rb}\right) - \frac{r}{b^2}.$$

Thus,

$$b^2\, E\left(-\frac{\partial^2 l}{\partial\lambda^2}\right) = \frac{1}{r^2}\sum_{j=1}^{*} t_j = \frac{1}{r},$$
$$b^2\, E\left(-\frac{\partial^2 l}{\partial\lambda\partial b}\right) = 0 \tag{7.4.5}$$

and

$$b^2\, E\left(-\frac{\partial^2 l}{\partial b^2}\right) = J - \frac{C^2}{r},$$

where J and C are as defined in section 7.3.1. Denote the MLE of λ by $\hat{\lambda}$ and J - C^2/r by

B. Then we obtain asymptotically

$$Var(\hat{\lambda}) = \frac{1}{E\left(-\dfrac{\partial^2 l}{\partial \lambda^2}\right)} = rb^2,$$

$$Var(\hat{b}) = \frac{1}{E\left(-\dfrac{\partial^2 l}{\partial b^2}\right)} = \frac{b^2}{B} \tag{7.4.6}$$

and

$Cov(\hat{\lambda},\hat{b}) = 0.$

Now, from the equation (7.4.2),

$$\begin{aligned} Var(\hat{u}) &= [\ Var(\hat{\lambda}) + C^2\ Var(\hat{b})\ ]/r^2 \\ &= b^2\ (1+ C^2/rB)/r \end{aligned} \tag{7.4.7}$$

and

$$\begin{aligned} Cov(\hat{u},\hat{b}) &= (\ Var\ (\hat{\lambda}) - r^2\ Var(\hat{u}) - C^2\ Var(\hat{b})\ )/2rC \\ &= -\ C\ Var(\hat{b})/r = -\ b^2\ C/rB\ . \end{aligned} \tag{7.4.8}$$

For complete sample, we have

$Var(\hat{\lambda}) = nb^2;$

$Var(\hat{b}) = 6b^2/n\pi^2;$

$Var(\hat{u}) = 6Kb^2/n\pi^2$

and

$$Cov(\hat{u},\hat{b}) = -\ 6(1-\gamma)b^2/n\pi^2. \tag{7.4.9}$$

These results for complete samples were also given by Nelson (1982, P. 337).

7.4.2 Extreme Value Regression Model

As obtained in section 7.4.1, the orthogonal parameter λ_p is given by, for p = 1,...,m,

$$\lambda_p = \sum_{s=1}^{m} A_{ps}\,\beta_s + bC_p\,. \tag{7.4.10}$$

By using the matrix notation, the expression in (7.4.10) can be written as

$$\lambda = A\beta + bC, \tag{7.4.11}$$

where $\lambda = (\lambda_1,\ldots,\lambda_m)'$; $\beta = (\beta_1,\ldots,\beta_m)'$ and the matrix $A = (A_{pq})_{m\times m}$ and the vector $C = (C_1,\ldots,C_m)'$ are as defined in section 7.3.2. In terms of the parameters $\lambda = (\lambda_1,\ldots,\lambda_m)'$ and b, the log likelihood (7.2.6) can now be written as

$$l(\lambda,b) = -r\log b + \sum_{j=1}^{r}\left(\frac{Y_j - Q_j\,\lambda}{b} + Q_j\,C\right) - \sum{}^{*} \exp\left(\frac{Y_j - Q_j\,\lambda}{b} + Q_j\,C\right) \tag{7.4.12}$$

where $Q_j\lambda = \sum Q_{js}\,\lambda_s$; $\qquad Q_{js} = \sum X_{jk}\,P_{ks}$, $\qquad s,k = 1,\ldots,m$, and the matrix $P = (P_{ks})$ is the inverse of the matrix A given in (7.4.11). From (7.4.12), we obtain, for p,q = 1,...,m,

$$\frac{\partial l}{\partial \lambda_p} = \frac{1}{b}\left\{\sum_{j=1}^{*} Q_{jp}\exp\left(\frac{Y_j - Q_j\lambda}{b} + Q_jC\right) - \sum_{j=1}^{r} Q_{jp}\right\},$$

$$\frac{\partial l}{\partial b} = \frac{1}{b}\left\{\sum_{j=1}^{*}\left(\frac{Y_j - Q_j\,\lambda}{b}\right)\exp\left(\frac{Y_j - Q_j\,\lambda}{b} + Q_j\,C\right) - \sum_{j=1}^{r}\left(\frac{Y_j - Q_j\,\lambda}{b}\right) - r\right\},$$

$$-\frac{\partial^2 l}{\partial\lambda_p\,\partial\lambda_q} = \frac{1}{b^2}\left\{\sum_{j=1}^{*} Q_{jp}\,Q_{jq}\,\exp\left(\frac{Y_j - Q_j\,\lambda}{b} + Q_j\,C\right)\right\},$$

$$-\frac{\partial^2 l}{\partial\lambda_p\,\partial b} = \frac{1}{b^2}\left\{\sum_{j=1}^{*} Q_{jp}\left(\frac{Y_j - Q_j\,\lambda}{b} + 1\right)\exp\left(\frac{Y_j - Q_j\,\lambda}{b} + Q_j\,C\right) - \sum_{j=1}^{r} Q_{jp}\right\}$$

and

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{b^2}\left\{\sum_{j=1}^{*}\left[\left(\frac{Y_j - Q_j\,\lambda}{b}\right)^2 + 2\left(\frac{Y_j - Q_j\,\lambda}{b}\right)\right]\exp\left(\frac{Y_j - Q_j\,\lambda}{b} + Q_j C\right)\right\}$$
$$+ \frac{2}{b^2}\sum_{j=1}^{r}\left(\frac{Y_j - Q_j\,\lambda}{b}\right) - \frac{r}{b^2}\,.$$

Thus,

$$b^2\,E\left(-\frac{\partial^2 l}{\partial\lambda_p\partial\lambda_q}\right) = P_{pq}\quad,\quad b^2\,E\left(-\frac{\partial^2 l}{\partial\lambda_p\partial b}\right) = 0$$

and

$$b^2 E\left(-\frac{\partial^2 l}{\partial b^2}\right) = B = J - C' A^{-1} C .$$

where P, J and C are as defined in section 7.3.2. As stated before, by inverting the Fisher information matrix of the maximum likelihood estimators $\hat{\lambda}_1,\ldots,\hat{\lambda}_m$ and $\hat{b}$, we obtain, asymptotically, $Var(\hat{b}) = b^2/B$; $Var(\hat{\lambda}) = b^2P^{-1} = Ab^2$; $Var(\beta) = b^2 [\, A^{-1} + A^{-1}CC'A^{-1}/B \,]$ and $Cov(\beta,\hat{b}) = b^2A^{-1}C/B$, where A and C are as defined in section 7.4.2.

When $r = n$, $Var(\hat{b})$, $Var(\beta)$ and $Cov(\beta,\hat{b})$ are almost numerically identical to the corresponding complete sample results of Lawless (1982, P.302).

7.5 INTERVAL ESTIMATION PROCEDURES FOR THE LOCATION AND SCALE PARAMETERS OF THE TWO PARAMETER EXTREME VALUE DISTRIBUTION

7.5.1 Intervals Based on Asymptotic Properties of the MLEs (AI)

Following the interval estimation procedure 2.16.1 described in chapter 2, the approximate $100(1-\alpha)\%$ confidence interval for the parameter b is given by

$$\hat{b} - \zeta \sqrt{Var}\,(\hat{b}) < b < \hat{b} + \zeta \sqrt{Var}\,(\hat{b}), \tag{7.5.1}$$

and that for the parameter u is given by

$$\hat{u} - \zeta \sqrt{Var}\,(\hat{u}) < u < \hat{u} + \zeta \sqrt{Var}\,(\hat{u}), \tag{7.5.2}$$

where ζ is an appropriate quantile of a standard normal random variable and $\hat{u}$ and $\hat{b}$ are the MLEs of u and b respectively. $Var(\hat{u})$ and $Var(\hat{b})$ are as defined in (7.4.7) and (7.4.6) respectively. Using the expressions for $Var(\hat{u})$ and $Var(\hat{b})$, we obtain

$$\hat{b}\ (\ 1\ -\ \zeta/\sqrt{B}\ ) < b < \hat{b}\ (\ 1\ +\ \zeta/\sqrt{B})\ . \tag{7.5.3}$$

and

$$\hat{u}\ -\ \zeta\ \hat{b}\ \sqrt{\frac{1+C^2/rB}{r}}\ < u < \hat{u}\ +\ \zeta\ \hat{b}\ \sqrt{\frac{1+C^2/rB}{r}}\ . \tag{7.5.4}$$

Denote these limits by b_{AL}, b_{AU}, u_{AL} and u_{AU} such that $b_{AL} < b < b_{AU}$ and $u_{AL} < u < u_{AU}$.

7.5.2 Intervals Based on Likelihood Score Corrected for Bias and Skewness (BI)

Consider the log likelihood function (7.4.3). Define

$$I_{\lambda\lambda}\ =\ -\ E\left(\frac{\partial^2 l}{\partial\lambda^2}\right);\ I_{\lambda b}\ =\ -\ E\left(\frac{\partial^2 l}{\partial\lambda\partial b}\right);\ I_{bb}\ =\ -\ E\left(\frac{\partial^2 l}{\partial b^2}\right)$$

and $I_{bb\lambda} = I_{bb} - I^2_{\lambda b} / I_{\lambda\lambda}$.

Suppose that we are interested in constructing confidence interval for the parameter b in presence of the nuisance parameter λ. Then from section 2.16.3, the adjusted score statistic

$$T_b\ =\ a\ \frac{\partial l}{\partial b}\ +\ c\ \frac{\partial l}{\partial\lambda},$$

where $a\ =\ (I_{bb\lambda})^{-1/2}$ and $c\ =\ -\ I_{\lambda b}\ I_{\lambda\lambda}^{-1}\ (I_{bb\ .\ \lambda})^{-1/2}$, has asymptotically standard

normal distribution. As discussed in section 2.16.3, the approximate 100(1-α)% confidence interval for b can be obtained by solving

$T_b - B(T_b) - K_3(b)\,(\zeta^2-1)/6 = \pm\,\zeta,$ (7.5.6)

where ζ is an appropriate quantile of a standard normal distribution, $B(T_b)$ is the bias of T_b of order $O(n^{-1})$ and $K_3(b)$ is the third cumulant of T_b of order $O(n^{-1/2})$. As shown in section 7.4.1, λ and b are orthogonal and hence $I_{\lambda b} = 0$. Thus,

$I_{bb.\lambda} = I_{bb}$, c = 0,

$$T_b = \frac{\partial l}{\partial b}\Big/\sqrt{I_{bb}}\ ,$$

$$B(T_b) = -\frac{I_{\lambda\lambda}^{-1}}{2\sqrt{I_{bb}}}\,E\left(\frac{\partial^3 l}{\partial b\,\partial\lambda^2}\right) + O(n^{-1}) \tag{7.5.7}$$

and

$$K_3(b) = E\left(\frac{\partial l}{\partial b}\right)^3 \Big/ \left(I_{bb}\right)^{3/2} = \left[2\,E\left(\frac{\partial^3 l}{\partial b^3}\right) + 3\,\frac{\partial I_{bb}}{\partial b}\right] \Big/ \left(I_{bb}\right)^{3/2} + O(n^{-1/2})\ . \tag{7.5.8}$$

In our context, we have

$$T_b = \left\{\sum_{j=1}^{*}\left(\frac{Y_j-\tilde{u}}{b}\right)\exp\left(\frac{Y_j-\tilde{u}}{b}\right) - \sum_{j=1}^{r}\left(\frac{Y_j-\tilde{u}}{b}\right) - r\right\}\Big/\sqrt{B},$$

$\tilde{u} = (\tilde{\lambda} - Cb)/r,$

$$\hat{\lambda} = \hat{\lambda}(\hat{b}) = rb \log \left[\frac{1}{r} \sum_{j=1}^{*} \exp\left(\frac{Y_j}{b} + \frac{C}{r} \right) \right],$$

$B(T_b) = (B)^{-1/2}$,

$K_3(b) = (2D - 6B)/B^{3/2}$

and

$D = F - 3\ CG/r + 6\ C^2/r + 2\ C^3/r^2$.

The quantities B and C are as defined in section 7.4.1 and the quantities F and G are, approximately,

$$F = \sum{}^{*} \left\{ \left(t_j \log t_j + \frac{3d_j}{2t_j} \right) \left((\log t_j)^2 + 6 \log t_j + 6 \right) \right\}$$

$$- 3 \sum_{j=1}^{r} \left(2 \log t_j - \frac{d_j}{t_j^2} \right) - 2r$$

and

$$G = \sum{}^{*} \left\{ \left(t_j \log t_j + \frac{d_j}{t_j} \right) \log t_j + \frac{d_j}{t_j} \right\} + 4C .$$

Denote the limits obtained from (7.5.6) by b_{BL} and b_{BU} such that $b_{BL} < b < b_{BU}$.

For the construction of confidence interval for u, it is necessary to deal with the log likelihood $l(u,b)$ given in (7.2.1). For this case,

$$I_{uu} = - E \left(\frac{\partial^2 l}{\partial u^2} \right); I_{ub} = - E \left(\frac{\partial^2 l}{\partial u \partial b} \right); I_{bb} = - E \left(\frac{\partial^2 l}{\partial b^2} \right)$$

and $I_{uu.b} = I_{uu} - I^2_{ub} / I_{bb}$.

Define the score statistic

$$T_u = a' \frac{\partial l}{\partial u} + c' \frac{\partial l}{\partial b},$$

where

$$a' = (I_{uu\,.\,b})^{-1/2} \text{ and } c' = -I_{ub}\,(I_{bb}\,)^{-1}\,(I_{uu\,.\,b}\,)^{-1/2}.$$

As we described above, the approximate $100(1-\alpha)\%$ confidence interval for u is obtained by solving

$$T_u - B(T_u\,) - \frac{K_3(u)}{6}(\zeta^2-1) = \pm\,\zeta\,, \tag{7.5.9}$$

where $B(T_u)$ is the bias of T_u and $K_3(u)$ is the third cumulant of T_u. Using the expressions of B, C, F and G given in this section we obtain,

$$I_{uu} = r/\bar{b}^2; \;\; I_{ub} = C/\bar{b}^2, \;\; I_{bb} = J/\bar{b}^2,$$

$$I_{uu.b} = \left(r - \frac{C^2}{J} \right) / \;\bar{b}^2, \;\; T_u = \frac{\partial l}{\partial u} \;/\; \bar{b}\,\sqrt{(r - \frac{C^2}{J})}\;,$$

$$\frac{\partial l}{\partial u} = \frac{1}{\bar{b}} \left\{ \sum_{j=1}^{*} \exp\left(\frac{Y_j - u}{\bar{b}} \right) - r \right\}, \tag{7.5.10}$$

$$\frac{\partial l}{\partial b} = \frac{1}{\bar{b}}\left\{\sum_{j=1}^{*}\left(\frac{Y_j-u}{\bar{b}}\right)\exp\left(\frac{Y_j-u}{\bar{b}}\right) - \sum_{j=1}^{r}\left(\frac{Y_j-u}{\bar{b}}\right) - r\right\}, \tag{7.5.11}$$

$$B(T_u) = -\left(G - \frac{CF}{J}\right)/\, 2\, J\sqrt{(r - \frac{C^2}{J})}\ , \tag{7.5.12}$$

and

$$K_3(u) = \left\{r - 3\frac{C(r+C)}{J} + 3\,(\frac{C}{J})^2\,(G-2C) + (\frac{C}{J})^3\,(3J-F)\right\}/3\,(r-\frac{C^2}{J})^{3/2}. \tag{7.5.13}$$

Note that equation (7.5.9) involves the nuisance parameter b which we replace by $\bar{b}(u)$, the maximum likelihood estimate of b for given u. This is obtained by setting

$\partial l/\partial b = 0.$ (7.5.14)

However, $\bar{b}(u)$ cannot be given explicitly from equation (7.5.14). Therefore the asymptotic 100(1-α)% confidence interval for u is obtained by solving (7.5.9) and (7.5.14) simultaneously. Note the quantities $B(T_u)$ and $K_3(u)$ do not depend on the parameters u and b. Denote $B(T_u) + K_3(u)\ (\zeta^2-1)/6$ by K_0. Then equation (7.5.9) can be written as

$T_u = K_0 \pm \zeta,$

which in turn can be written as

$$\sum_{j=1}^{*}\exp\left(\frac{Y_j-u}{\bar{b}}\right) - r = (\,K_0 \pm \zeta\,)\sqrt{r-\frac{C^2}{J}}\ . \tag{7.5.15}$$

Now, from (7.5.10) we obtain

$$u = \tilde{b} \log \left[\frac{\sum_{j=1}^{*} \exp(Y_j / \tilde{b})}{r + (K_0 \pm \zeta) \sqrt{r - \frac{C^2}{J}}} \right]. \tag{7.5.16}$$

Putting (7.5.16) in (7.5.15) yields two equations which involve only b. The two solutions obtained from the resultant equations are then put in (7.5.16) to produce the confidence limits for u. Denote these limits by u_{BL} and u_{BU} such that $u_{BL} < u < u_{BU}$.

7.5.3 Intervals Based on Likelihood Ratio (LI)

This procedure has been reviewed by Lawless (1982) for constructing confidence intervals for the extreme value distribution location and scale parameters with censored data. Now, the log likelihood function in terms of the parameters u and b is given in (7.2.1). Then, as described in section 2.16.2, the likelihood ratio statistic (LR_b) for testing b is given by $LR_b = 2\ [l(\hat{u},\hat{b}) - l(\tilde{u},b)]$, which is asymptotically distributed as chi squared with one degree of freedom, where $l(\hat{u},\hat{b})$ is the unrestricted maximum log likelihood function and $l(\tilde{u},b)$ is the restricted maximum log likelihood function. $\tilde{u} = \tilde{u}(b)$ is a function of b obtained by setting $\partial l/\partial u = 0$. In our context, we obtain

$$LR_b = 2 \left[r \log(b/\hat{b}) + \sum_{j=1}^{r} \left(\frac{Y_j - \hat{u}}{\hat{b}} - \frac{Y_j - \tilde{u}}{b} \right) \right], \tag{7.5.17}$$

where

$$\tilde{u} = b \log \left[\frac{1}{r} \sum_{j=1}^{*} \exp(Y_j / b) \right]. \tag{7.5.18}$$

The b values that satisfy

$$LR_b = \chi^2_{(1-\alpha)}(1), \tag{7.5.19}$$

where $\chi^2_{(1-\alpha)}(1)$ is the $(1-\alpha)$th chi square quantile with one degree of freedom, are the approximate $100(1-\alpha)\%$ confidence limits for b. Denote these limits by b_{LL} and b_{LU} such that $b_{LL} < b < b_{LU}$. Note that the expression in LR_b involves only the parameter b when we replace $\tilde{u}$ by its estimate given in (7.5.18). Thus the equation (7.5.19) can be readily solved iteratively for the values of b.

For the construction of confidence interval for u, the necessary likelihood ratio statistic is $LR_u = 2\,[l(\hat{u},\hat{b}) - l(u,\tilde{b})]$, where $\tilde{b} = \tilde{b}(u)$ is the value of b that maximizes the log likelihood function $l(u,b)$ for a given value of u. This value for b can be obtained by setting $\partial l/\partial b = 0$. In our context, we obtain

$$LR_u = 2\left[r \log(b/\hat{b}) - r + \sum_{j=1}^{r}\left(\frac{Y_j-\hat{u}}{\hat{b}} - \frac{Y_j-u}{\tilde{b}} \right) + \sum_{j=1}^{*} \exp\left(\frac{Y_j-u}{\tilde{b}} \right) \right] \tag{7.5.20}$$

and

$$\frac{\partial l}{\partial b} = \left\{ \sum_{j=1}^{*} \left(\frac{Y_j-u}{b} \right) \exp\left(\frac{Y_j-u}{b} \right) - \sum_{j=1}^{r} \left(\frac{Y_j-u}{b} \right) - r \right\} / \, b \, . \tag{7.5.21}$$

Thus the u values that satisfy

$$LR_u = \chi^2_{(1-\alpha)}(1) \tag{7.5.22}$$

subject to $\partial l/\partial b = 0$, are the approximate $100(1-\alpha)\%$ confidence limits for u. Denote these limits by u_{LL} and u_{LU} such that $u_{LL} < u < u_{LU}$. Note that the value $\tilde{b} = \tilde{b}(u)$ cannot be expressed explicitly as a function of u, from the equation (7.5.21). Therefore, to obtain the confidence limits for u, we need to solve the equations (7.5.22) and (7.5.21) simultaneously.

7.5.4 Intervals Based on Adjusted Likelihood Ratio (DI)

Diciccio, Field and Fraser (1990) derived a procedure based on an approximation to the distribution of the signed root of the likelihood ratio statistic for constructing confidence intervals for the location- scale parameters of the extreme value distribution having pdf (2.13.5). Suppose the log likelihood function in terms of u and b is given by (7.2.1). Define

$V_1 = (u-\hat{u})/\hat{b}$, $V_2 = \log(b/\hat{b})$ and the quantities

$A_j = (Y_j-\hat{u})/\hat{b}$, j = 1,...,r, whose distributions are parameter free. Then the log likelihood (7.2.1) reduces to

$$l(V_1,V_2) = -rV_2 + \sum_{j=1}^{r} P_j - \sum_{j=1}^{*} \exp(P_j) , \tag{7.5.23}$$

where $P_j = (A_j-V_1)\exp(-v_2)$, j = 1,...,r. Since $\hat{u}$ and $\hat{b}$ are the MLEs of u and b respectively it is clear that $l(V_1,V_2)$ attains its maximum at $\mathbf{V} = (V_1,V_2)' = (0,0)' = \mathbf{0}$. So we denote the unrestricted maximum log likelihood function by $l(0,0)$. If the parameter b is of interest then the likelihood ratio (Λ_b) can be given as

$\Lambda_b = 2\,[l(0,0) - l(V_1,V_2)]$, where $\tilde{V}_1$ is the value of V_1 that maximizes the log likelihood $l(\mathbf{V})$ for a given value of V_2. The value for $\tilde{V}_1$ can be obtained by setting $\partial l/\partial V_1 = 0$.

Define

$$SR_b = \begin{cases} -\sqrt{\Lambda_b} \ , & b<\hat{b} \\ \sqrt{\Lambda_b} \ , & b>\hat{b} \end{cases} .$$

Now, the marginal tail probability of v_2 is, approximately, given by

$$p(V_2 \le v_2) = \Phi(SR_b) + \phi(SR_b)[s_b^*] + O(n^{-3/2}) . \tag{7.5.24}$$

where Φ and ϕ are respectively the distribution function and the density function of a standard normal random variable and

$$S_b^* = \frac{1}{SR_b} + \frac{|I^0|^{1/2}}{l_b(\bar{V}_1,V_2)\ (-l_{uu}(\bar{V}_1,V_2))^{1/2}} ,$$

$$l_b(\bar{V}_1,V_2) = \left. \frac{\partial l}{\partial V_2} \right|_{V_1=\bar{V}_1}$$

$$- l_{uu}(\bar{V}_1,V_2) = - \left. \frac{\partial^2 l}{\partial V_1^2} \right|_{V_1=\bar{V}_1}$$

and

I^0 is the observed information matrix. In our context, we obtain

$$\Lambda_b = 2 \left\{ rV_2 - \sum_{j=1}^{r} (\bar{P}_i - A_i) \right\}, \ \bar{P}_i = (A_i - \bar{V}_1)e^{-v_2} ,$$

$$\bar{V}_1 = e^{V_2} \log \left[\frac{1}{r} \sum_{j=1}^{*} \exp(A_i \, e^{-V_2}) \right],$$

$$l_b(\bar{V}_1, V_2) = \sum_{j=1}^{*} \bar{P}_i \, e^{\bar{P}_i} - \sum_{j=1} \bar{P}_i - r,$$

$$l_{uu}(\bar{V}_1, V_2) = \left(\sum_{j=1}^{*} e^{\bar{P}_i} \right) e^{-2V_2},$$

$$I^0_{V:uu} = - \left. \frac{\partial^2 l}{\partial V_1 \partial V_1} \right|_{V=0} = r,$$

$$I^0_{V:ub} = - \left. \frac{\partial^2 l}{\partial V_1 \partial V_2} \right|_{V=0} = \sum_{j=1}^{*} A_i \, e^{A_i}$$

and

$$I^0_{V:bb} = - \left. \frac{\partial^2 l}{\partial V_2 \partial V_2} \right|_{V=0} = \sum_{j=1}^{*} A_i^2 \, e^{A_i} + r .$$

The approximate 100(1-α)% lower and upper confidence limits $V^b{}_L$ and $V^b{}_U$ of V_2 can be obtained by setting the expression (7.5.24) to $\alpha/2$ and to (1-$\alpha/2$) respectively. It is easily shown that the approximate 100(1-α)% confidence interval for the parameter b is given by

$$\hat{b} \, e^{v_L^b} < b < \hat{b} \, e^{v_U^b} .$$

Denote these limits by b_{DL} and b_{DU} such that $b_{DL} < b < b_{DU}$.

As discussed above, for the construction of confidence interval for the parameter u, the approximate marginal distribution of the pivotal V_1 is given by

$$P(V_1 \le v_1) = \Phi(SR_u) + \phi(SR_u)\,(S^*_u) + O(n^{-3/2}), \tag{7.5.25}$$

where

$$\Lambda_u = 2\,[l(0,0) - l(V_1,\tilde{V}_2)]\ ,$$

$$SR_u = \begin{cases} -\sqrt{\Lambda_u}\ , & u<\hat{u} \\ \sqrt{\Lambda_u}\ , & u>\hat{u} \end{cases}\ ,$$

$$S^*_u = \frac{1}{SR_u} + \frac{|I^0|^{1/2}}{l_u(V_1,\tilde{V}_2)\ [-\ l_{bb}(V_1,\tilde{V}_2)]^{1/2}}\ ,$$

$$l_u(V_1,\tilde{V}_2) = \left.\frac{\partial l}{\partial V_1}\right|_{V_2=\tilde{V}_2}\ ,$$

$$l_{bb}(V_1,\tilde{V}_2) = \left.\frac{\partial^2 l}{\partial V_2^2}\right|_{V_2=\tilde{V}_2}\ ,$$

and $\tilde{V}_2 = \tilde{V}_2(V_1)$ is the value of V_2 that maximizes the log likelihood function $l(V)$ for a given value of v_1. In our context, we obtain

$$\Lambda_u = 2\left[\ r(\tilde{V}_2-1) + \sum_{j=1}^{r}(A_i-\tilde{P}_i) + \sum_{j=1}^{*}\ e^{\tilde{P}_i}\right],$$

where $\tilde{P}_i = (A_i-V_1)\ \exp(-\tilde{V}_2)$, $l_u(V_1,\tilde{V}_2) = e^{-\tilde{V}_2}\left(\sum_{j=1}^{*}\ e^{\tilde{P}_i} - r\right)$ and

$$- l_{bb}(V_1,\tilde{V}_2) = \sum_{j=1}^{*} \tilde{P}_i \, e^{\tilde{P}_i} + r.$$

$\tilde{V}_2$ is obtained by setting $\partial l/\partial V_2 = 0$; that is

$$\sum_{j=1}^{*} \tilde{P}_i \, e^{\tilde{P}_i} - \sum_{j=1}^{r} \tilde{P}_i - r = 0. \tag{7.5.26}$$

So the approximate 100(1-α)% lower and upper confidence limits $V^u{}_L$ and $V^u{}_U$ of V_1 can be obtained by setting the expression (7.5.25) to $\alpha/2$ and to $(1-\alpha/2)$ respectively. Note that the expression (7.5.25) involves the estimate $\tilde{V}_2$, which cannot be obtained explicitly from (7.5.26). Therefore, to obtain the limits stated above, we need to solve the above appropriate equations with the equation (7.5.26) simultaneously. Thus, the approximate 100(1-α)% confidence interval for the parameter u obtained from the pivotal V_1, approximately, is

$\hat{u} + \hat{b}\, V^u{}_L < u < \hat{u} + \hat{b}\, V^u{}_U$. Denote these limits by u_{DL} and u_{DU} such that $u_{DL} < u < u_{DU}$.

7.5.5 Simulation Study

A simulation study was conducted to examine the behaviour of the confidence intervals AI, BI, LI and DI in terms of average lengths, tail probabilities and coverage probabilities. The two parameter extreme value distributed random variables, with u = 0.15 and b = 0.9, for various combinations of (n,r), where n is the sample size and r is the number of failures, were generated via the IMSL (1987) Weibull random number generator RNWIB. Each experiment was based on 2000 replications. For all the procedures, we computed average lengths of the confidence intervals, the tail probabilities and the coverage probabilities based on 2000 samples using nominal levels $\alpha = 0.10$ and 0.05. Results for the parameters b and u are reported in Tables 7.2 and 7.3 respectively.

Results

We now discuss the results for the parameter b. From Table 7.2 we can see that the procedure AI is inaccurate even for large samples such as $n = 40$. The procedure LI provides coverage closer to nominal under no censoring or very light censoring, except in the small sample situation, for example, $n = 10$. Moreover, the procedures LI and AI have always asymmetric tail probabilities. The procedure BI, in general, produces satisfactory results except in small sample situations (for example $n \leq 20$) in which the coverage probability is greater than the nominal. Overall, the procedure DI yields nearly symmetric tail probabilities and desired coverage even for small sample sizes and heavy censoring. The average interval lengths in ascending order correspond to AI, LI, DI and BI.

Results in Table 7.3, for the confidence intervals for the parameter u indicate that all the procedures provide desired coverage and nearly symmetric tail probabilities for large samples under no censoring. When the degree of censoring increases the procedures LI and AI yield smaller coverage than desired and have asymmetric tail probabilities. The behaviour of the procedures BI and DI are, in general, similar. However, the procedure DI tends to yield more symmetric tail probabilities and accurate coverage for small sample sizes under heavy censoring. Considering the average interval lengths based on all procedures, the procedure AI provides the shortest confidence intervals and the procedure BI yields the longest confidence intervals. However, as the degree of censoring decreases, the lengths based on all procedures tend to be closer.

Considering the length of the intervals and coverage and tail probabilities, the procedure DI is the preferred method for small samples with the degrees of censoring considered. The performance of the procedure BI is similar to that of the procedure DI for moderate to large samples, but it gives interval lengths slightly longer than those of DI. The procedure LI is good for complete samples, where it provides desired coverage and shorter interval lengths than those of DI and BI. The procedure AI is unsatisfactory based on coverage probabilities even though it always yields the shortest lengths.

7.6 INTERVAL ESTIMATION PROCEDURES FOR THE PARAMETERS OF EXTREME VALUE REGRESSION MODEL

7.6.1 Intervals Based on Asymptotic Properties of the MLEs (AI)

As the performance of this procedure in the two parameter model was poor, we do not consider this method in the regression situation.

7.6.2 Intervals Based on Likelihood Score Corrected for Bias and Skewness (BI)

Consider the log likelihood function $l(\beta,b)$ given in (7.2.7). Suppose that the scale parameter b is of interest and the regression parameters $\beta_1,\ldots,\beta_m$ are treated as nuisance parameters. As described in section 2.16.2, denote

$$I_{bb} = -E\left(\frac{\partial^2 l}{\partial b^2}\right); \quad I_{b\beta_p} = -E\left(\frac{\partial^2 l}{\partial b \partial \beta_p}\right); \quad I_{\beta\beta} = -E\left(\frac{\partial^2 l}{\partial \beta \partial \beta'}\right),$$

where $I_{\beta\beta}$ is of order mxm, and $I_{bb.\beta} = I_{bb} - I_{b\beta}\, I_{\beta\beta}^{-1}\, I_{\beta b}$.

The adjusted score is

$$T_b = \frac{\partial l}{\partial b} - I_{b\beta}\, I_{\beta\beta}^{-1}\, \frac{\partial l}{\partial \beta},$$

where

$$\frac{\partial l}{\partial \beta} = \left(\frac{\partial l}{\partial \beta_1}, \ldots, \frac{\partial l}{\partial \beta_m}\right)'.$$

Define $f = (f_1, \ldots, f_m)' = I_{b\beta}\, I_{\beta\beta}^{-1}$. Then the statistic T_b can be written as

$$T_b = \frac{\partial l}{\partial b} - f.\frac{\partial l}{\partial \beta} \tag{7.6.1}$$

To the order $O(n^{-1})$, the bias of T_b is given by (Bartlett, 1955; Levin and Kong, 1990)

$$B(T_b) = -\frac{1}{2}\, trace\left\{ I_{\beta\beta}^{-1} \left[E\left(\frac{\partial^3 l}{\partial b \partial \beta \partial \beta'}\right) + 2\, \frac{\partial I_{b\beta}}{\partial \beta} \right] \right\} + \frac{1}{2}\, trace\left[I_{\beta\beta}^{-1}\, M \right], \tag{7.6.2}$$

where

$$M_j = \left\{ E \left(\frac{\partial^3 l}{\partial \beta_j \partial \beta \partial \beta'} \right) + 2 \, \frac{\partial I_{\beta\beta}}{\partial \beta_j} \right\} I_{\beta\beta}^{-1} \, I_{\beta b}.$$

The third cumulant of T_b is given, for s, t, q = 1,...,m, as

$$\begin{aligned} K_3(b) &= 2 \, E \left(\frac{\partial^3 l}{\partial b^3} \right) + 3 \, \frac{\partial I_{bb}}{\partial b} \\ &\quad -3 \sum_s f_s \left[2 \, E \left(\frac{\partial^3 l}{\partial b^2 \partial \beta_s} \right) + 2 \, \frac{\partial I_{b\beta_s}}{\partial b} + \frac{\partial I_{bb}}{\partial \beta_s} \right] \\ &\quad +3 \sum_s \sum_t f_s f_t \left[2 \, E \left(\frac{\partial^3 l}{\partial \beta_s \partial \beta_t \partial b} \right) + \frac{\partial I_{\beta_s \beta_t}}{\partial b} + \frac{\partial I_{b\beta_t}}{\partial \beta_s} + \frac{\partial I_{b\beta_s}}{\partial \beta_t} \right] \\ &\quad - \sum_s \sum_t \sum_q f_s f_t f_q \left[2 \, E \left(\frac{\partial^3 l}{\partial \beta_s \partial \beta_t \partial \beta_q} \right) + \frac{\partial I_{\beta_t \beta_q}}{\partial \beta_s} + \frac{\partial I_{\beta_q \beta_s}}{\partial \beta_t} + \frac{\partial I_{\beta_s \beta_t}}{\partial \beta_q} \right] \end{aligned} \tag{7.6.3}$$

In case of failure censored data from extreme value distribution, for s, t, q = 1,...,m, the necessary quantities to compute T_b, $B(T_b)$ and $K_3(b)$ are as follows:

$$\frac{\partial l}{\partial b} = \frac{1}{b} \left[\sum_{j=1}^{*} \left(\frac{Y_j - X_j \, \beta}{b} \right) \exp \left(\frac{Y_j - X_j \, \beta}{b} \right) - \sum_{j=1}^{r} \left(\frac{Y_j - X_j \, \beta}{b} \right) - r \right],$$

$$\frac{\partial l}{\partial \beta_s} = \frac{1}{b} \left[\sum_{j=1}^{*} X_{js} \, \exp \left(\frac{Y_j - X_j \, \beta}{b} \right) - \sum_{j=1}^{r} X_{js} \right],$$

$$I_{\beta\beta} = A \ , \ I_{\beta b} = C \ , \ I_{bb} = J \ ,$$

$$\frac{\partial I_{bb}}{\partial \beta_s} = 0 \ , \ \frac{\partial I_{\beta_t \beta_q}}{\partial \beta_s} = 0 \ , \ \frac{\partial I_{b\beta_t}}{\partial \beta_s} = 0 \ ,$$

$$b^3 \frac{\partial I_{\beta_s \beta_t}}{\partial b} = -2 A_{st} \ , \ b^3 \frac{\partial I_{b\beta_s}}{\partial b} = -2 C_s \ , \ b^3 \frac{\partial I_{bb}}{\partial b} = -2 J \ ,$$

$$b^3 E \left(\frac{\partial^3 l}{\partial \beta_s \partial \beta_t \partial \beta_q} \right) = \sum_{j=1}^{*} X_{js} X_{jt} X_{jq} \ t_j \ ,$$

$$b^3 E \left(\frac{\partial^3 l}{\partial \beta_s \partial \beta_t \partial b} \right) = 2 A_{st} + \sum_{j=1}^{*} X_{js} X_{jt} \left(t_j \log t_j + \frac{d_j}{2t_j} \right) ,$$

$$b^3 E \left(\frac{\partial^3 l}{\partial \beta_s \partial b^2} \right) = \sum_{j=1}^{*} X_{js} \{ t_j \ ((\log t_j)^2 + 4 \log t_j + 2) + \frac{d_j}{t_j} (3 + \log t_j) \} - 2 \sum_{j=1}^{r} X_{js}$$

and

$b^3 E(\partial^3 l / \partial b^3) = F$,

where the terms A_{st}, C_s and J, for s,t = 1,...,m, are as given in (7.3.6) and F is as in section 7.5.2. As stated earlier, the statistic T_b corrected for bias and skewness is asymptotically distributed as Normal with mean zero and variance $I_{bb.\beta}$. Thus, the approximate 100(1-α)% confidence interval for b can be obtained by solving

$$\frac{T_b}{\sqrt{I_{bb.\beta}}} - \frac{B(T_b)}{\sqrt{I_{bb.\beta}}} - \frac{K_3(b)\ (\zeta^2-1)}{6(I_{bb.\beta})^{3/2}} = \pm\ \zeta\ . \tag{7.6.4}$$

Since the quantities $B(T_b)/(I_{bb.\beta})^{1/2}$ and $K_3(b)/(I_{bb.\beta})^{3/2}$ are parameter free we denote the expression $B(T_b)/(I_{bb.\beta})^{1/2} + K_3(b)(\zeta^2 - 1)/[6(I_{bb.\beta})^{3/2}]$ by K_0. Equation (7.6.4) then reduces to

$$T_b = (K_0 \pm \zeta)\ (I_{bb.\beta})^{1/2}. \tag{7.6.5}$$

Note that the term T_b in (7.6.5) involves the nuisance parameters $\beta_1,...,\beta_m$. These parameters can be replaced by their ML estimators obtained by setting $\partial l/\partial\beta_s = 0$, s = 1,...,m.

Now we consider the simple linear regression model (m = 2). we here deal with only one covariate X, and $\beta = (\beta_1, \beta_2)'$. Thus, we have

$$T_b = \frac{\partial l}{\partial b} - f_1 \frac{\partial l}{\partial\beta_1} - f_2 \frac{\partial l}{\partial\beta_2}\ , \tag{7.6.6}$$

$$\frac{\partial l}{\partial b} = \frac{1}{b}\left\{\sum_{j=1}^{*}\left(\frac{Y_j - X_j\beta}{b}\right)\exp\left(\frac{Y_j - X_j\beta}{b}\right) - \sum_{j=1}^{r}\left(\frac{Y_j - X_j\beta}{b}\right) - r\right\}, \tag{7.6.7}$$

$$\frac{\partial l}{\partial \beta_1} = \frac{1}{b}\left\{ \sum_{j=1}^{*} \exp\left(\frac{Y_j - X_j\beta}{b} \right) - r \right\}, \tag{7.6.8}$$

$$\frac{\partial l}{\partial \beta_2} = \frac{1}{b}\left\{ X_j \exp\left(\frac{Y_j - X_j\,\beta}{b} \right) - \sum_{j=1}^{r} X_j \right\}, \tag{7.6.9}$$

where $X_j\beta = \beta_1 X_{j1} + \beta_2 X_{j2}$ with $X_{j1} = 1$, $j = 1,\ldots,r$,

$f_1 = (A_{22}\, C_1 - A_{12}\, C_2)/[\, r\, A_{22} - (A_{12})^2\,]$,

and

$f_2 = (r\, C_2 - A_{12}\, C_1)/[\, r\, A_{22} - (A_{12})^2\,]$.

Bias of T_b is given by

$$B(T_b) = \frac{A_{22}\,(\Delta_1) + r\,(\Delta_2) - 2\,A_{12}\,(\Delta_3)}{2[r\,A_{22} - (A_{12}\,)^2]\,b}, \tag{7.6.10}$$

where

$\Delta_1 = (f_1\text{-}2)\, r + f_2\, A_{12} - C_1$,

$\Delta_2 = (f_1\text{-}2)\, A_{22} + f_2\, A^0 - C^0$,

$\Delta_3 = (f_1\text{-}2)\, A_{12} + f_2\, A_{22} - C_2$,

$$C^0 = \sum_{j=1}^{*} X_j^2\,(t_j \log t_j + \frac{d_j}{2\,t_j}) \text{ and } A^0 = \sum_{j=1}^{*} X_j^3 t_j\,.$$

The third cumulant of T_b is

$$K_3(b) = \{ 2\,(F - f_1^3\, r - f_2^3\, A^0) - 6\,[\, J + f_1(Y^0 + 2C_1) + f_2(Y^{00} + 2C_2)\,]$$

$$+ 6\,[\, f_1^2\,(r + C_1) + f_2^2\,(A_{22} + C^0) - f_1^2\, f_2\, A_{12}\,] \qquad (7.6.11)$$

$$- 6\,[\, f_1\, f_2^2\, A_{22} - 2\, f_1\, f_2\,(A_{12} + C_2)\,]\,\} \,/\, b^3\,,$$

where

$$Y^0 = \sum_{j=1}^{*} \{\, t_j\,(\log t_j\,)^2 + \frac{d_j}{t_j}\,(1 + \log t_j\,)\,\}$$

and

$$Y^{00} = \sum_{j=1}^{*} X_j\,\{\, t_j\,(\log t_j\,)^2 + \frac{d_j}{t_j}\,(1 + \log t_j\,)\,\}\,,$$

and

$I_{bb.\beta} = J - (\, F_1 + F_2 - 2F_3)/\, 2(\, r\, A_{22} - (A_{12})^2)$, where $F_1 = A_{22}\, C_1{}^2$, $F_2 = r\, C_2{}^2$ and $F_3 = A_{12}$ $C_1\, C_2$.

Note that equation (7.6.6) involves the parameter of interest b as well as the nuisance parameters β_1 and β_2. So equation (7.6.5) needs to be solved subject to the constraints $\partial l/\partial\beta_1 = 0$ and $\partial l/\partial\beta_2 = 0$. However, some simplification is possible. From equation (7.6.8), that is, by setting $\partial l/\partial\beta_1 = 0$, we obtain β_1 in terms of b and β_2 as

$$\beta_1 = b\,\log\left\{\frac{1}{r}\sum_{j=1}^{*}\exp\left(\frac{Y_j - X_j\,\beta_2}{b}\right)\right\}.$$

Substituting this value of β_1 in $\partial l/\partial\beta_2 = 0$ and in (7.6.5) we obtain

$$\frac{\sum_{j=1}^{*} X_j \exp\left(\frac{Y_j - X_j \beta_2}{b}\right)}{\sum_{j=1}^{*} \exp\left(\frac{Y_j - X_j \beta_2}{b}\right)} - \frac{1}{r} \sum_{j=1}^{r} X_j = 0 \tag{7.6.12}$$

and

$$\frac{1}{b} \left[\frac{r \sum_{j=1}^{*} Y_j \exp\left(\frac{Y_j - X_j \beta_2}{b}\right)}{\sum_{j=1}^{*} \exp\left(\frac{Y_j - X_j \beta_2}{b}\right)} - \sum_{j=1}^{*} Y_j - rb \right] = (K_0 \pm \zeta) \sqrt{I_{bb.\beta}} \ . \tag{7.6.13}$$

The equations (7.6.12) and (7.6.13) involve b and β_2. Simultaneous solutions of these equations yield the desired approximate 100(1-α)% confidence limits for b. Denote these limits as b_{BL} and b_{BU} such that $b_{BL} < b < b_{BU}$.

For constructing the confidence interval for a regression parameter β_p, p = 1,...,m, denote $\theta = \beta_p$ and $\phi = (\phi_1,...,\phi_m)' = (\beta_1,..\beta_{p-1},\beta_{p+1},...,\beta_m,b)'$. $I_{\theta\theta}$, $I_{\theta\phi}$, $I_{\phi\phi}$ and $I_{\theta\theta.\phi}$ are as defined in section 2.16.2. Define the statistic

$$T_\theta = \frac{\partial l}{\partial \theta} - \sum_{s=1}^{m} g_s \frac{\partial l}{\partial \phi_s}. \tag{7.6.14}$$

where g_s, s = 1,...,m, are the elements of $I_{\theta\phi} I^{-1}{}_{\phi\phi}$. As discussed above, the approximate 100(1-α)% confidence interval for θ (=β_p) can be obtained by solving

$$\frac{T_\theta}{\sqrt{I_{\theta\theta.\phi}}} - \frac{B(T_\theta)}{\sqrt{I_{\theta\theta.\phi}}} - \frac{K_3(\theta)\ (\zeta^2 - 1)}{6\ (I_{\theta\theta.\phi})^{3/2}} = \pm \zeta \ , \tag{7.6.15}$$

where $B(T_\theta)$ and $K_3(\theta)$ are, respectively, the bias and the third cumulant of T_θ. Further, we denote the parameter free quantity $B(T_\theta)/(I_{\theta\theta.\phi})^{1/2} + K_3(\theta)(\zeta^2 - 1)/[6(I_{\theta\theta.\phi})^{3/2}]$ by K'_0. Thus equation (7.6.15) becomes

$$T_\theta = (K'_0 \pm \zeta)\,(I_{\theta\theta.\phi})^{1/2}. \qquad (7.6.16)$$

We can see that the term T_θ in (7.6.16) depends on the vector of parameters $\phi = (\phi_1,\ldots,\phi_m)' = (\beta_1,\ldots,\beta_{p-1},\beta_{p+1},\ldots,\beta_m,b)'$, which can be replaced by their ML estimators for given β_p. Thus, for a given β_p the MLEs of these parameters can be obtained by setting $\partial l/\partial\beta_s = 0$, $s = 1,\ldots,m$; $s \neq p$ and $\partial l/\partial b = 0$. Hence, to obtain the approximate $100(1-\alpha)\%$ confidence interval for the regression parameter β_p we need to solve equations (7.6.16), $\partial l/\partial\beta_s = 0$, $s = 1,\ldots,m$; $s \neq p$, and $\partial l/\partial b = 0$ simultaneously.

Now, we will discuss the results for a simple linear regression model ($m = 2$). In this situation we have $\beta = (\beta_1, \beta_2)'$ and one regressor variable X. Then $X_j\beta = \beta_1 X_{j1} + \beta_2 X_{j2}$, with $X_{j1} = 1$, $j = 1,\ldots,r$. Suppose the regression parameter β_2 is of interest. Then $\theta = \beta_2$; $\phi = (\phi_1,\phi_2)' = (\beta_1,b)'$ and

$$T_\theta = \partial l/\partial\theta - g_1\, \partial l/\partial\phi_1 - g_2\, \partial l/\partial\phi_2,$$

where $\partial l/\partial\theta = \partial l/\partial\beta_2$, $\partial l/\partial\phi_1 = \partial l/\partial\beta_1$ and $\partial l/\partial\phi_2 = \partial l/\partial b$, which are given in equations (7.6.9), (7.6.8) and (7.6.7) respectively. The terms g_1 and g_2 are obtained as

$$g_1 = (J\, A_{12} - C_1\, C_2)/[\, R\, J - (C_1)^2\,]$$

and

$$g_2 = (R\, C_2 - A_{12}\, C_1)/[\, RJ - (C_1)^2\,].$$

To the order $O(n^{-1})$, the bias of the statistic T_θ is given by

$$B(T_\theta) = \{\, J\, \Delta_1 + r\, \Delta_2 - 2\, C_1\, \Delta_3\, \}/\{\, 2\,(r\, J - C^2_1)\, \}b\,,$$

where

$\Delta^1{}_1 = g_1 r + g_2 (2r+C_1) - A_{12}$,

$\Delta^1{}_2 = g_1 Y^0 + g_2 (F-4J) - (Y^{00} + 2 T^{00} - 2 A_{12})$,

$\Delta^1{}_3 = g_1 C_1 + g_2 (Y^0 + 2 C_1) - C_2$,

$T^{00} = X_1 + ... + X_r$.

Now, the third cumulant of T_θ is given by,

$$K_3(\theta) = \{ 2 [A^0 - f_1^3 r - g_2^3 (F-3 J)] - 6 [g_1 A_{22} + g_2 (A_{22} + C^0)]$$
$$+ 6 [g_1^2 A_{12} + g_2^2 (Y^{00} + 2C_2) + 2g_1 g_2 (A_{12} + C_2)]$$
$$- 6 [g_1^2 g_2 (r + C_1) + g_1 g_2^2 (Y^0 + 2C_1)] \} b^{-3}$$

The quantity $I_{\theta\theta.\phi}$ is given by

$$I_{\theta\theta.\phi} = [J A_{12}^2 + r C_2^2 - 2 A_{12} C_1 C_2]/[r J - C_1^2].$$

We can see that the expression for T_θ involves the nuisance parameters β_1 and b. Thus, equation (7.6.16) needs to be solved for the parameter of interest θ (= β_2) subject to the constraints $\partial l/\partial\beta_1 = 0$ and $\partial l/\partial b = 0$. By setting equation (7.6.8) to zero we obtain

$$\beta_1 = b \log \left\{ \frac{1}{r} \sum_{j=1}^{*} \exp\left(\frac{Y_j - \beta_2 X_j}{b} \right) \right\}.$$

Substituting this in (7.6.16) and $\partial l/\partial b = 0$ we obtain

$$\frac{1}{\tilde{b}}\left\{\frac{r\sum_{j=1}^{*} X_j \exp\left(\frac{Y_j-\beta_2 X_j}{\tilde{b}}\right)}{\sum_{j=1}^{*} \exp\left(\frac{Y_j-\beta_2 X_j}{\tilde{b}}\right)} - \sum_{j=1}^{r} X_j\right\} = (K_0 \pm \zeta)\sqrt{I_{\theta\theta.\phi}} \qquad (7.6.17)$$

and

$$\frac{r\sum_{j=1}^{*}\left(\frac{Y_j - \beta_2 X_j}{\tilde{b}}\right)\exp\left(\frac{Y_j - \beta_2 X_j}{\tilde{b}}\right)}{\sum_{j=1}^{*} \exp\left(\frac{Y_j - \beta_2 X_j}{\tilde{b}}\right)} - \sum_{j=1}^{r}\left(\frac{Y_j - \beta_2 X_j}{\tilde{b}}\right) - r = 0 .$$

(7.6.18)

Note that $\tilde{b}$ can not be expressed explicitly as a function of β_2 from $\partial l/\partial b = 0$. Therefore, the approximate 100(1-α)% confidence interval for β_2 needs to be computed by solving the equations (7.6.17) and (7.6.18) simultaneously. Denote these limits by β_{BL} and β_{BU} such that $\beta_{BL} < \beta < \beta_{BU}$.

7.6.3 Intervals Based on Likelihood Ratio (LI)

Consider the likelihood function given by (7.2.2). As we discussed in section 7.5, for the construction of confidence interval for b, the likelihood ratio statistic (LR_b) is defined as $LR_b = 2\,[\, l(\hat{\beta},\hat{b}) - l(\tilde{\beta},b)\,]$, where $l(\hat{\beta},\hat{b})$ is the unrestricted maximum log likelihood function and $l(\tilde{\beta},b)$ is the restricted maximum log likelihood function. For the extreme value distribution with failure censored data, we obtain

$$LR_b = 2\,[\, r \log (b/\hat{b}) + \sum_{j=1}^{r} (\frac{Y_j - X_j\hat{\beta}}{\hat{b}} - \frac{Y_j - X_j\tilde{\beta}}{b})\,] , \qquad (7.6.19)$$

which is asymptotically distributed as chi-square with one degree of freedom. Note that β can not be expressed as a function of b by setting $\partial l/\partial\beta_p = 0$, p = 1,.....,m, where the derivative $\partial l/\partial\beta_p$ is as defined in (7.2.8). So, the b values that satisfy

$$LR_b = \chi^2_{(1-\alpha)}(1) \tag{7.6.20}$$

subject to $\partial l/\partial\beta_p = 0$, p =1,...,m, are the approximate 100(1-α)% confidence limits for b. Denote the limits by b_{LL} and b_{LU} such that $b_{LL} < b < b_{LU}$.

For constructing the confidence interval for β_p, p = 1,...,m, the necessary likelihood ratio LR_β is given by $LR_\beta = 2\,[l(\hat{\beta},\hat{b}) - l(\tilde{\beta},\tilde{b})]$, where $\tilde{\beta} = \tilde{\beta}(\beta_p)$ and $\tilde{b} = \tilde{b}(\beta_p)$ are the values that maximize the log likelihood function for a given value of β_p. These values can not be expressed explicitly by setting $\partial l/\partial\beta_s = 0$, s = 1,...,m; s $\neq$ p and $\partial l/\partial b = 0$. So, to obtain 100(1-$\alpha$)% confidence interval for β_p we need to solve $LR_\beta = \chi^2_{(1-\alpha)}(1)$, $\partial l/\partial\beta_s = 0$, s = 1,...,m; s $\neq$ p and $\partial l/\partial b = 0$ simultaneously. In our context, we have

$$LR_\beta = 2\left[r\log(\tilde{b}/\hat{b}) - r + \sum_{j=1}^{r}\left(\frac{Y_j - X_j\,\hat{\beta}}{\hat{b}} - \frac{Y_j - X_j\,\tilde{\beta}}{\tilde{b}}\right) + \sum_{j=1}^{*}\exp\left(\frac{Y_j - X_j\,\tilde{\beta}}{\tilde{b}}\right)\right] \tag{7.6.21}$$

and $\partial l/\partial\beta_s = 0$, s = 1,...,m, and $\partial l/\partial b = 0$ are as given in equations (7.2.8) and(7.2.9). Denote the limits obtained by β_{LL} and β_{LU} such that $\beta_{LL} < \beta < \beta_{LU}$.

7.6.4 Intervals Based on Adjusted Likelihood Ratio (DI)

As described in section 7.5.4, we define $V_p = (\beta_p - \hat{\beta}_p)/\hat{b}$, p = 1,...,m; $V_{m+1} = \log(b/\hat{b})$ and $A_j = (Y_j - X_j\,\hat{\beta})/\hat{b}$, j = 1,...,r. Denote $V = (V_1,...,V_m,V_{m+1})'$ and $A = (A_1,...,A_r)'$. Now, for given A, the log likelihood (7.2.2) can be written as

$$l(V) = -r V_{m+1} + \sum_{j=1}^{r} P_i - \sum_{j=1}^{*} \exp(P_j), \tag{7.6.23}$$

where, for j = 1,...,r,

$$P_j = (A_j - \sum_{s=1}^{m} X_{js} V_s) \exp(-V_{m+1}).$$

Suppose the scale parameter b is of interest. Then the associated pivotal quantity is V_{m+1}. Following the procedure discussed in section 2.16.4, the approximate marginal distribution of V_{m+1} is given by

$$P(V_{m+1} \leq v_{m+1}) = \Phi(SR_{m+1}) + \phi(sR_{m+1}) [S^*_{m+1}] + O(n^{-3/2}), \tag{7.6.24}$$

where Φ and ϕ are as defined in section 7.5.4, and

$$SR_{m+1} = \begin{cases} -\sqrt{LR_{m+1}} \ , & b < \hat{b} \\ \sqrt{LR_{m+1}} \ , & b > \hat{b}, \end{cases}$$

$$S^*_{m+1} = \frac{1}{SR_{m+1}} + \frac{|I^0 |^{1/2}}{l_{m+1} (\tilde{V}(V_{m+1})) \ | \ I^* (\tilde{V}(V_{m+1})) \ |^{1/2}},$$

$$l_{m+1} (\tilde{V} (V_{m+1})) = \left. \frac{\partial l}{\partial V_{m+1}} \right|_{V = \tilde{V} (V_{m+1})} .$$

I^0 is the observed information matrix of order (m+1)x(m+1).

$I^*(\tilde{V}(V_{m+1}))$ is a sub matrix of I^0 which is of order mxm.

$|I^0|^{1/2}$ and $|I^*|^{1/2}$ are the square roots of the determinants of the matrices I^0 and I^*

respectively. LR_{m+1} is the likelihood ratio statistic for testing the scale parameter b. Thus, the approximate 100(1-α)% confidence limits for the pivotal V_{m+1} are obtained by setting the expression in (7.6.24) to $\alpha/2$ and $1-\alpha/2$ subject to $\partial l/\partial V_p = 0$, p =1,...,m. Denote these limits by $V_L{}^b$ and $V_U{}^b$ such that $V_L{}^b < V_{m+1} < V_U{}^b$. From the pivotal V_{m+1}, we obtain the approximate 100(1-α)% confidence limits for b as $\hat{b}\exp(V_L{}^b) < b < \hat{b}\exp(V_U{}^b)$. Define these limits by b_{DL} and b_{DU} such that $b_{DL} < b < b_{DU}$.

In case of simple linear regression (m = 2), the required pivotal quantities are $V_1 = (\beta_1 - \hat{\beta}_1)/\hat{b}$, $V_2 = (\beta_2 - \hat{\beta}_2)/\hat{b}$ and $V_3 = \log(b/\hat{b})$. Thus the likelihood ratio statistic for setting confidence interval for b in terms of pivotals is given by

$$LR_3 = 2\left[\, r\,V_3 - \sum_{j=1}^{r} (\tilde{P}_j - A_j)\,\right],$$

where $\tilde{P}_j = (A_j - \tilde{V}_1 - X_j\tilde{V}_2)\exp(-V_3)$. The elements of the observed information matrix and of the matrix I^* are as follows:

$$I_{11}^0 = -\left.\frac{\partial^2 l}{\partial V_1^2}\right|_{V=0} = r,$$

$$I_{12}^0 = -\left.\frac{\partial^2 l}{\partial V_1 \partial V_2}\right|_{V=0} = \sum_{j=1}^{r} X_j,$$

$$I_{22}^0 = -\left.\frac{\partial^2 l}{\partial V_2^2}\right|_{V=0} = \sum_{j=1}^{*} X_j^2 \exp(A_j),$$

$$I_{13}^0 = -\left.\frac{\partial^2 l}{\partial V_1 \partial V_3}\right|_{V=0} = \sum_{j=1}^{r} A_j + r ,$$

$$I_{23}^0 = -\left.\frac{\partial^2 l}{\partial V_2 \partial V_3}\right|_{V=0} = \sum_{j=1}^{*} X_j A_j \exp(A_j) ,$$

$$I_{33}^0 = -\left.\frac{\partial^2 l}{\partial V_3^2}\right|_{V=0} = \sum_{j=1}^{*} A_j^2 \exp(A_j) + r ,$$

$$I_{11}^{*}(V_3) = -\left.\frac{\partial^2 l}{\partial V_1^2}\right|_{V=\tilde{V}(V_3)} = \left(\sum_{j=1}^{*} \exp(\tilde{P}_j\right)\exp(-2V_3) ,$$

$$I_{12}^{*}(V_3) = -\left.\frac{\partial^2 l}{\partial V_1 \partial V_2}\right|_{V=\tilde{V}(V_3)} = \left(\sum_{j=1}^{*} X_j \exp(\tilde{P}_j)\right)\exp(-2V_3) ,$$

$$I_{22}^{*}(V_3) = -\left.\frac{\partial^2 l}{\partial V_2^2}\right|_{V=\tilde{V}(V_3)} = \left(\sum_{j=1}^{*} X_j^2 \exp(\tilde{P}_j\right)\exp(-2V_3)$$

and

$$l_3(\tilde{V}(V_3)) = \left.\frac{\partial l}{\partial V_3}\right|_{V=\tilde{V}(V_3)} = \sum_{j=1}^{*} \tilde{P}_j \exp(\tilde{P}_j - \sum_{j=1}^{r} \tilde{P}_j - r .$$

For the construction of the confidence interval for a regression coefficient β_p, p = 1,...,m, the associated pivotal is V_p. Then the approximate marginal distribution of V_p is given by

$$P(V_p \le v_p) = \Phi(\, SR_p \,) + \phi(\, SR_p \,)\, [\, S_p^* \,] + O(n^{-3/2}\,), \tag{7.6.25}$$

where Φ and ϕ are as defined in section 7.5.4. The quantity SR_p is defined as

$$SR_p = \begin{cases} -\sqrt{LR_p} \ , & \beta_p < \beta_p \\ \sqrt{LR_p} \ , & \beta_p > \beta_p \end{cases} ,$$

where $LR_p = 2\,[l(0) - l(\bar{V}(V_p))]$ is the likelihood ratio statistic for testing β_p. The quantity $l(0)$ is the unrestricted maximum log likelihood function and the quantity $l(\bar{V}(V_p))$ is the maximum log likelihood function for a given value of V_p. Expression for the term S^*_p in (7.6.25) is given by

$$S_p^* = \frac{1}{SR_p} + \frac{|I^0\ |^{1/2}}{l_p\,(V_p\,)\ |I^*\ \bar{V}(V_p\,)\ |^{1/2}}$$

and

$$l_p(V_p) = \left.\frac{\partial l}{\partial V_p}\right|_{V = \bar{V}(\,V_p\,)} ,$$

where

the matrix I^0 is the observed information matrix as defined earlier,

the matrix I^* is a sub matrix of I^0 corresponding to $V_1,\ldots,V_{p-1},V_{p+1},\ldots,V_m,V_{m+1}$, and

the term $\bar{V}(V_p)$ can be expressed by setting $\partial l/\partial V_s = 0$, $s = 1,\ldots,(m+1)$; $s \neq p$.

Now, the approximate 100(1-α)% confidence limits for β_p, p = 1,...,m, are obtained by setting the expression in (7.6.25) to $\alpha/2$ and 1-$\alpha/2$ subject to $\partial l/\partial V_s = 0$, s = 1,...,(m+1); s $\neq$ p. Denote the limits by V_L^p and V_U^p such that $V_L^p < V_p < V_U^p$. From the definition of pivotal V_p, the approximate 100(1-α)% confidence interval for β_p is given as $\hat{\beta}_p + \hat{b} V_L^p < \beta_p < \hat{\beta}_p + \hat{b} V_U^p$. Denote these limits by β_{DL}^p and β_{DU}^p such that $\beta_{DL}^p < \beta_p < \beta_{DU}^p$. When m = 2, we obtain

$$LR_2 = 2 \left[r (\tilde{V}_3 - 1) + \sum_{j=1}^{r} (A_j - \tilde{P}_j) + \sum_{j=1}^{*} \exp(\tilde{P}_j) \right],$$

$$\tilde{P}_j = (A_j - \tilde{V}_1 - V_2 X_j) \exp(-\tilde{V}_3),$$

$$l_2(V_2) = \left.\frac{\partial l}{\partial V_2}\right|_{V = \tilde{V}(V_2)} = \left(\sum_{j=1}^{*} X_j \exp(\tilde{P}_j) - \sum_{j=1}^{r} X_j \right) \exp(-\tilde{V}_3),$$

$$I_{11}^{*}(V_2) = -\left.\frac{\partial^2 l}{\partial V_1^2}\right|_{V = \tilde{V}(V_2)} = \left(\sum_{j=1}^{*} \exp(\tilde{P}_j) \right) \exp(-2\tilde{V}_3),$$

$$I_{12}^{*}(V_2) = -\left.\frac{\partial^2 l}{\partial V_1 \partial V_3}\right|_{V = \tilde{V}(V_2)} = \left(\sum_{j=1}^{*} \tilde{P}_j + r \right) \exp(-\tilde{V}_3)$$

and

$$I_{22}^{*}\ (V_2) = -\left.\frac{\partial^2 l}{\partial V_3{}^2}\right|_{V = \tilde{V}(V_2)} = \sum_{j=1}^{*} \tilde{P}_j^2 \exp(\tilde{P}_j) + r\ .$$

Table 7.2: Average lengths(AL), lower tail(L), upper tail(U) and coverage(C) probabilities(%) of the confidence intervals AI, BI, LI and DI for b under failure censoring. Based on 2000 samples.
u = 0.15; b = 0.9.

		α = 10.0				α = 5.0			
n,r		AL	L	U	C	AL	L	U	C
10,10	AI	0.677	0.4	20.2	79.4	0.807	0.1	15.2	84.7
	BI	1.050	3.9	3.0	93.0	1.335	1.6	1.1	97.3
	LI	0.749	2.4	10.7	86.9	0.924	0.9	5.7	93.3
	DI	0.898	4.7	4.5	90.8	1.114	2.6	2.4	95.0
10,5	AI	0.998	0.3	21.5	78.2	1.215	0.0	13.7	86.3
	BI	2.096	3.7	2.7	93.6	2.691	1.3	1.7	97.0
	LI	1.206	1.4	17.3	81.2	1.554	0.7	10.4	88.9
	DI	1.949	4.5	5.1	90.4	2.583	2.1	2.6	95.2
20,20	AI	0.496	0.9	13.6	85.5	0.591	0.2	9.5	90.3
	BI	0.602	4.0	4.0	92.0	0.731	2.0	2.0	96.0
	LI	0.522	2.5	8.1	89.4	0.633	1.1	4.2	94.7
	DI	0.568	4.1	4.4	91.4	0.690	2.1	2.0	95.9
20,10	AI	0.767	0.2	21.2	78.6	0.914	0.0	16.2	83.8
	BI	1.240	3.3	2.9	93.7	1.549	1.6	1.1	97.3
	LI	0.862	1.9	10.7	87.4	1.067	0.7	5.5	93.7
	DI	1.059	4.1	4.1	91.7	1.319	2.2	2.1	95.7
40,40	AI	0.356	1.6	10.1	88.3	0.427	0.4	6.4	93.2
	BI	0.392	4.7	4.3	91.0	0.472	2.1	2.2	95.7
	LI	0.367	3.2	6.5	90.3	0.442	1.4	3.8	94.8
	DI	0.383	4.6	4.6	90.7	0.460	2.1	2.4	95.5
40,30	AI	0.456	1.5	12.5	86.0	0.544	0.2	8.6	91.2
	BI	0.519	4.6	4.2	91.2	0.626	2.3	1.9	95.8
	LI	0.473	3.2	8.0	88.8	0.570	1.1	4.1	94.8
	DI	0.502	4.9	4.7	90.3	0.606	2.3	2.4	95.3
40,20	AI	0.577	0.9	14.9	84.1	0.688	0.1	10.6	89.2
	BI	0.713	3.8	4.4	91.8	0.847	2.1	2.2	95.7
	LI	0.612	2.5	9.0	88.5	0.743	1.2	5.1	93.7
	DI	0.674	4.2	5.1	90.7	0.819	2.3	2.9	94.8

Table 7.3: Average lengths(AL), lower tail(L), upper tail(U) and coverage(C) probabilities(%) of the confidence intervals AI, BI, LI and DI for u under failure censoring. Based on 2000 samples.
u = 0.15; b = 0.9.

		α = 10.0				α = 5.0			
n,r		AL	L	U	C	AL	L	U	C
10,10	AI	0.914	7.2	7.2	85.6	1.089	4.9	5.0	90.2
	BI	1.108	5.1	4.1	90.8	1.494	2.7	1.0	96.3
	LI	0.987	5.4	6.8	87.8	1.216	3.3	3.9	92.8
	DI	1.089	4.9	4.8	90.3	1.350	2.8	2.1	95.1
10,5	AI	1.009	2.1	16.9	81.0	1.072	0.5	11.8	87.7
	BI	2.471	4.1	3.6	92.3	3.398	1.3	1.9	96.8
	LI	1.499	4.0	14.3	81.7	1.951	2.3	8.9	88.8
	DI	2.256	4.8	5.0	90.2	3.016	2.6	2.6	94.8
20,20	AI	0.669	5.9	6.5	87.6	0.798	3.6	3.9	92.5
	BI	0.726	4.9	5.0	90.1	0.900	2.6	2.3	95.1
	LI	0.695	5.0	6.2	88.8	0.841	3.0	3.5	93.5
	DI	0.727	4.8	5.1	90.1	0.881	2.6	2.9	94.5
20,10	AI	0.935	1.6	17.7	80.7	1.114	0.8	13.9	85.3
	BI	1.342	4.9	3.9	91.2	1.610	2.4	1.7	95.9
	LI	1.057	3.3	10.3	86.4	1.317	1.7	5.3	93.0
	DI	1.258	4.5	4.6	90.9	1.576	2.3	2.4	95.3
40,40	AI	0.484	5.2	4.8	90.0	0.576	2.5	2.9	94.6
	BI	0.502	4.5	4.1	91.4	0.615	2.0	2.4	95.6
	LI	0.493	4.4	4.8	90.8	0.592	2.0	2.9	95.1
	DI	0.503	4.3	4.1	91.6	0.605	2.0	2.4	95.6
40,30	AI	0.525	4.1	7.6	88.3	0.626	1.9	4.8	93.3
	BI	0.576	4.4	4.0	91.6	0.709	1.8	1.6	96.6
	LI	0.545	4.2	5.3	90.5	0.660	1.8	2.8	95.4
	DI	0.565	4.4	4.3	91.3	0.685	2.0	1.9	96.1
40,20	AI	0.701	2.1	13.1	84.7	0.835	0.7	9.8	90.0
	BI	0.824	4.3	4.0	91.7	0.992	2.3	2.0	95.7
	LI	0.745	3.4	7.1	89.5	0.908	1.9	3.8	94.3
	DI	0.808	4.4	4.1	91.4	0.986	2.4	1.9	95.7

CHAPTER 8

INTERVAL ESTIMATION FOR THE PARAMETERS OF EXTREME VALUE MODELS UNDER TIME CENSORING

8.1 INTRODUCTION

The previous chapter presents various procedures to set approximate confidence intervals for the parameters of the extreme value models with failure censored data. In this chapter we deal with the same problem under time censoring. For singly time censored data from extreme value distribution, Meeker and Nelson (1974, 1977) propose an approximate method based on the MLEs and their asymptotic variances which are obtained by using a table produced by Monte Carlo simulations. Lawless (1982) suggested an approximate method based on likelihood ratio statistic for setting confidence intervals for the parameters of the extreme value distribution as well as the extreme value regression model. The extreme value model in regression analysis based on time censored data has received attention by many authors, recently, due to its widespread use in the area of life testing and reliability (Ostrouchov and Meeker, 1988; Vander Weil and Meeker, 1990; Bugarghis, 1988 and Doganaksoy and Scheme, 1991). Ostrouchov and Meeker (1988) studied LR based intervals for the extreme value parameters and quantiles based on interval censored data. Vander Weil and Meeker (1990) examined the likelihood based intervals in the accelerated life tests using the inverse power law model. Bugarghis (1988) investigated the bias and mean squares of the scale and the regression parameters through simulations. Doganaksoy and Scheme (1991) investigate improvements of LR based intervals and the corrected signed root of the LR statistic and Bartlett correction to the LR statistic discussed by Diciccio (1988).

The MLEs of the parameters of both extreme value distribution and extreme value regression model are presented in section 8.2. In section 8.3, we construct the expected Fisher information matrices for the MLEs of the parameters. The asymptotic variance-covariance matrices are provided in section 8.4. In section 8.5, the confidence interval procedures are derived for the parameters of the two parameter extreme value model under time censoring. These procedures are then compared in terms of coverage probabilities, tail probabilities and average lengths. Relevant derivations and simulation study for constructing confidence intervals for the parameters of the extreme value regression model are given in section 8.6.

8.2 MAXIMUM LIKELIHOOD ESTIMATION

8.2.1 Two Parameter Extreme Value Distribution

Denote the observed lifetime as T_i and the fixed censoring time L_i for the ith experimental item in a random sample of n items. Observations are of the form $t_i = \text{Min}(T_i,L_i)$, $i = 1,\ldots,n$. The random variables T_i's are assumed to have a Weibull distribution with pdf (2.13.4) or equivalently $Y_i = \log t_i$, $i = 1,\ldots,n$ has an extreme value distribution with pdf (2.13.5). Denote $\eta_i = \log L_i$, $i = 1,\ldots,n$. For convenience we define

$$\delta_i = \begin{cases} 1 \ , & t_i = T_i \\ 0 \ , & t_i = L_i \end{cases} .$$

Following section 2.12.2, the likelihood for the ith item under time censoring is given by

$$L_i(u,b) = \left\{ \frac{1}{b} \exp\left[\left(\frac{Y_j - u}{b} \right) - \exp\left(\frac{Y_j - u}{b} \right) \right] \right\}^{\delta_i} \left\{ \exp\left[-\exp\left(\frac{\eta_i - u}{b} \right) \right] \right\}^{1-\delta_i} .$$

(8.2.1)

Thus, the log likelihood for the ith item is

$$l_i(u,b) = \delta_i \left\{ \left(\frac{Y_i - u}{b} \right) - \log b - \exp\left(\frac{Y_i - u}{b} \right) \right\} + (1-\delta_i) \left\{ -\exp\left(\frac{\eta_i - u}{b} \right) \right\} .$$

(8.2.2)

Let r be the number of failures, then $\sum_{i=1}^{n} \delta_i = r$.

Then the likelihood for the entire data can be written as

$$l(u,b) = \sum_{i=1}^{n} l_i = -r \log b + \sum_{i=1}^{r} \left(\frac{Y_i - u}{b} \right) - \sum_{i=1}^{n} \exp\left(\frac{Y_i - u}{b} \right) \qquad (8.2.3)$$

Note that in the last term of (8.2.3), there are r observed life times and (n-r) censored times, and also we can see that (8.2.3) is of the form (7.2.1). Taking derivatives of l(u,b) with respect to u and b, and equating to zero yields the maximum likelihood equations

$$\frac{\partial l}{\partial u} = \frac{1}{b}\left\{ \sum_{i=1}^{n} \exp\left(\frac{Y_i - u}{b}\right) - r \right\} = 0 \tag{8.2.4}$$

and

$$\frac{\partial l}{\partial b} = \frac{1}{b}\left\{ \sum_{i=1}^{n} \left(\frac{Y_i - u}{b}\right) \exp\left(\frac{Y_i - u}{b}\right) - \sum_{i=1}^{r} \left(\frac{Y_i - u}{b}\right) - r \right\} = 0. \tag{8.2.5}$$

The MLEs $\hat{u}$ and $\hat{b}$ of u and b can be obtained by solving equations (8.2.4) and (8.2.5) simultaneously. However, calculations can be reduced by eliminating the parameter u from equation (8.2.5). From (8.2.4), we have $u = b \log\left[\frac{1}{r}\sum_{i=1}^{n} \exp(Y_i/b)\right]$

Substituting the value of u in (8.2.5), we obtain

$$\frac{\sum_{i=1}^{n} Y_i \exp(Y_i/\hat{b})}{\sum_{i=1}^{n} \exp(Y_i/\hat{b})} - \frac{1}{r}\sum_{i=1}^{r} Y_i - \hat{b} = 0. \tag{8.2.6}$$

The non-linear equation (8.2.6) involves $\hat{b}$ only. It can be solved iteratively for $\hat{b}$, the MLE of b, with the use of computer. Once $\hat{b}$ is obtained, the MLE, $\hat{u}$, of u is given by

$$\hat{u} = \hat{b} \log\left[\frac{1}{r}\sum_{i=1}^{n} \exp(Y_i/\hat{b})\right]. \tag{8.2.7}$$

8.2.2 Extreme Value Regression Model

As defined in section 7.2.2, assume that the location parameter u is a function of m covariates $X = (X_1, \ldots, X_m)'$ such that $u(X) = \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_m X_m$ with $X_1 = 1$, where the regression vector of coefficients $\beta = (\beta_1, \ldots, \beta_m)'$ is unknown and is to be estimated from the available sample data. Then the log likelihood (8.2.3) can be written as

$$l(\beta,b) = -r \log b + \sum_{i=1}^{r} \left(\frac{Y_i - X_i \beta}{b} \right) - \sum_{i=1}^{n} \exp\left(\frac{Y_i - X_i \beta}{b} \right), \tag{8.2.8}$$

where Y_i is either a log life time or a log censoring time; r is the number of failures. The estimating equations for the MLEs of the parameters are obtained from (8.2.8) as, for p = 1,...,m,

$$\frac{\partial l}{\partial \beta_p} = \frac{1}{b} \left\{ \sum_{i=1}^{n} X_{ip} \exp\left(\frac{Y_i - X_i \beta}{b} \right) - \sum_{i=1}^{r} X_{ip} \right\} = 0 \tag{8.2.9}$$

and

$$\frac{\partial l}{\partial b} = \frac{1}{b} \left\{ \sum_{i=1}^{n} \left(\frac{Y_i - X_i \beta}{b} \right) \exp\left(\frac{Y_i - X_i \beta}{b} \right) - \sum_{i=1}^{r} \left(\frac{Y_i - X_i \beta}{b} \right) - r \right\} = 0. \tag{8.2.10}$$

Maximum likelihood estimators $\hat{\beta}_p$, p = 1,...,m, and $\hat{b}$ follow as the simultaneous solutions of the above (m+1) equations given in (8.2.9) and (8.2.10).

8.3 FISHER INFORMATION MATRIX

8.3.1 Two Parameter Extreme Value Distribution

From the log likelihood (8.2.2), the negative mixed partial derivatives of the sample likelihood are obtained as follows:

$$-\frac{\partial^2 l}{\partial u^2} = \frac{1}{b^2} \sum_{i=1}^{n} \left\{ \delta_i \exp\left(\frac{y_i - u}{b} \right) + (1-\delta_i) \exp\left(\frac{\eta_i - u}{b} \right) \right\},$$

$$-\frac{\partial^2 l}{\partial u\, \partial b} = \frac{1}{b^2} \sum_{i=1}^{n} \delta_i \left[\left(\frac{Y_i - u}{b} \right) \exp\left(\frac{Y_i - u}{b} \right) + \exp\left(\frac{Y_i - u}{b} \right) - 1 \right]$$

$$+ \frac{1}{b^2} \sum_{i=1}^{n} (1-\delta_i) \left[\left(\frac{\eta_i - u}{b} \right) \exp\left(\frac{\eta_i - u}{b} \right) + \exp\left(\frac{\eta_i - u}{b} \right) \right]$$

and

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{b^2} \sum_{i=1}^{n} \delta_i \left\{ \left[\left(\frac{Y_i - u}{b} \right)^2 + 2 \left(\frac{Y_i - u}{b} \right) \right] \exp\left(\frac{Y_i - u}{b} \right) - 2 \left(\frac{Y_i - u}{b} \right) - 1 \right\}$$

$$+ \frac{1}{b^2} \sum_{i=1}^{n} (1-\delta_i) \left\{ \left[\left(\frac{\eta_i - u}{b} \right)^2 + 2 \left(\frac{\eta_i - u}{b} \right) \right] \exp\left(\frac{\eta_i - u}{b} \right) \right\}.$$

Now, we define $Z_i = (Y_i - u)/b$ and $k_i = (\eta_i - u)/b$. Then, we have

$$-\frac{\partial^2 l}{\partial u^2} = \frac{1}{b^2}\sum_{i=1}^{n}\left\{\ \delta_i \exp(Z_i) + (1-\delta_i)\exp(k_i)\right\},$$

$$-\frac{\partial^2 l}{\partial u\,\partial b} = \frac{1}{b^2}\sum_{i=1}^{n}\left\{\begin{array}{c}\delta_i\left[\,(Z_i+1)\exp(Z_i) - 1\,\right]\\ +\,(1-\delta_i)\left[(k_i+1)\exp(k_i)\right]\end{array}\right\}$$

and

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{b^2}\sum_{i=1}^{n}\left\{\begin{array}{c}\delta_i\left[\,(Z_i^2 + 2\,Z_i)\exp(Z_i) - 2\,Z_i - 1\,\right]\\ +\,(1-\delta_i)\left[\,(k_i^2 + 2\,k_i)\exp(k_i)\,\right]\end{array}\right\}.$$

We obtain expected values of the negative of the mixed partial derivatives of the likelihood as

$$A = -\,b^2\ E\left(\frac{\partial^2 l}{\partial u^2}\right) = \sum_{i=1}^{n}\left\{1 - e^{-e^{k_i}}\right\},$$

$$C = -\,b^2\ E\left(\frac{\partial^2 l}{\partial u\,\partial b}\right) = \sum_{i=1}^{n}\left\{\int_0^{e^{k_i}} V\,(\log V)\,e^{-V}\,dV + k_i\,e^{k_i - e^{k_i}}\right\}$$

and

$$B = -\,b^2\,E\left(\frac{\partial^2 l}{\partial b^2}\right) = \sum_{i=1}^{n}\left\{\begin{array}{c}\int_0^{e^{k_i}} V(\log V)^2\,e^{-V}\,dV + k_i^2\,e^{k_i - e^{k_i}}\\ +\,1 - e^{-e^{k_i}}\end{array}\right\}.$$

The expressions for the terms A, B and C are also given in Lawless (1982) and Nelson

(1982). Thus, the expected Fisher information matrix is given by

$$I = \begin{bmatrix} -E\left(\frac{\partial^2 l}{\partial u^2}\right) & -E\left(\frac{\partial^2 l}{\partial u \partial b}\right) \\ E\left(\frac{\partial^2 l}{\partial u \partial b}\right) & -E\left(\frac{\partial^2 l}{\partial b^2}\right) \end{bmatrix} = \frac{1}{b^2}\begin{bmatrix} A & C \\ C & B \end{bmatrix}. \tag{8.3.1}$$

8.3.2 Extreme Value Regression Model

From the likelihood (8.2.8), we obtain

$$-\frac{\partial^2 l}{\partial \beta_p \, \partial \beta_q} = \frac{1}{b^2} \sum_{i=1}^{n} \left\{ X_{ip} \; X_{iq} \; \left[\; \delta_i \; \exp(Z_i) + (1-\delta_i) \exp(k_i) \right] \right\},$$

$$-\frac{\partial^2 l}{\partial \beta_p \, \partial b} = \frac{1}{b^2} \sum_{i=1}^{n} \left\{ X_{ip} \left[\; \delta_i \; (Z_i \, e^{Z_i} + e^{Z_i} - 1) + (1-\delta_i) \, k_i \, e^{k_i} \right] \right\}$$

and

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{b^2} \sum_{i=1}^{n} \left\{ \begin{array}{c} \delta_i \left[(Z_i^2 + 2 Z_i) \, e^{Z_i} - 2 Z_i - 1 \right] \\ + (1-\delta_i) \left[(k_i^2 + 2 k_i) \, e^{k_i} \right] \end{array} \right\},$$

where $Z_i = (Y_i - X_i \beta)/b$ and $k_i = (\eta_i - X_i \beta)/b$, $i = 1,\ldots,n$. We obtain, for $p, q = 1,\ldots,n$,

$$A_{pq} = -b^2 E\left(\frac{\partial^2 l}{\partial\beta_p\,\partial\beta_q}\right) = \sum_{i=1}^{n} X_{ip}\; X_{iq}\;(1 - e^{-e^{k_i}}),$$

$$C_p = -b^2 E\left(\frac{\partial^2 l}{\partial\beta_p\,\partial b}\right) = \sum_{i=1}^{n} X_{ip}\;\left\{\int_0^{e^{k_i}} V\,(\log V)\,e^{-V}\,dV + k_i^2\; e^{k_i - e^{k_i}}\right\}$$

and

$$B = -b^2 E\left(\frac{\partial^2 l}{\partial b^2}\right) = \sum_{i=1}^{n}\left\{\begin{array}{c}\int_0^{e^{k_i}} V\,(\log V)^2\,e^{-V}\,dV + k_i^2\; e^{k_i - e^{k_i}} \\ + 1 - e^{-e^{k_i}}\end{array}\right\}.$$

Thus, the expected Fisher Information matrix I is of order (m+1)x(m+1) and of the partitioned form, for p, q = 1,...,m, as

$$I = \begin{bmatrix} -E\left(\dfrac{\partial^2 l}{\partial\beta_p\,\partial\beta_q}\right) & -E\left(\dfrac{\partial^2 l}{\partial\beta_p\,\partial b}\right) \\ -E\left(\dfrac{\partial^2 l}{\partial\beta_p\,\partial b}\right) & -E\left(\dfrac{\partial^2 l}{\partial b^2}\right)\end{bmatrix} = \begin{bmatrix} A_{11} & . & . & . & A_{1m} & C_1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{ml} & . & . & . & A_{mm} & C_m \\ C_1 & . & . & . & C_m & B \end{bmatrix}.$$

(8.3.2)

These results concur with the results of Nelson (1978; 1982, P. 342). The integrals involved in the expressions A, B and C in section 8.3 can be easily evaluated numerically by using the routines such as DQDAGS from the IMSL library.

8.4 ASYMPTOTIC VARIANCE-COVARIANCE OF THE MLEs

8.4.1 Two Parameter Extreme Value Distribution

Following the definition 2.4, the asymptotic variance-covariance of the MLEs of the parameters u and b can be obtained by inverting the expected Fisher information matrix I given in (8.3.1), and thus $Var(\hat{u}) = B\ b^2/(AB-C^2)$, $Var(\hat{b}) = A\ b^2/(AB-C^2)$ and $Cov(\hat{u},\hat{b}) = -\ C\ b^2/(AB-C^2)$, where the quantities A, B and C are as defined in section 8.3.1.

8.4.2 Extreme Value Regression Model

As discussed in section 8.4.1, the inverse of the expected Fisher information matrix I given in (8.3.2) implies

$Var(\hat{\beta}) = b^2\ [\ A^{-1} + A^{-1}CC'A^{-1}/B\ ]$, $Var(\hat{b}) = b^2\ /(\ B - C'A^{-1}C\ )$ and $Cov(\ \hat{\beta},\hat{b}) = b^2\ A^{-1}C/(B-C'A^{-1}C)$, where the matrix A, the vector C and the term B are as defined in section (8.3.2).

Note that in the two parameter situation the terms A, B and C represent scalars, while in the regression situation A is a mxm matrix, C is a mx1 vector and B is a scalar.

8.5 INTERVAL ESTIMATION PROCEDURES FOR THE LOCATION AND SCALE PARAMETERS OF THE EXTREME VALUE DISTRIBUTION

8.5.1 Intervals Based on Asymptotic Properties of the MLEs (AI)

As stated in section 2.16.1, the approximate $100(1-\alpha)\%$ confidence intervals for the scale parameter b is

$$\hat{b} - \zeta \sqrt{Var(\hat{b})} < b < \hat{b} + \zeta \sqrt{Var(\hat{b})}, \tag{8.5.1}$$

and for the location parameter u is

$$\hat{u} - \zeta \sqrt{Var(\hat{u})} < u < \hat{u} + \zeta\sqrt{Var(\hat{u})}, \tag{8.5.2}$$

where ζ is an appropriate quantile of a standard Normal random variate. Substituting the expressions for Var($\hat{b}$) and Var($\hat{u}$) from section 8.4.1 in (8.5.1) and (8.5.2) yields

$$\hat{b}\left(1 - \zeta \sqrt{\frac{A}{AB - C^2}} \right) < b < \hat{b}\left(1 + \zeta \sqrt{\frac{A}{AB - C^2}} \right) \tag{8.5.3}$$

and

$$\hat{u} - \zeta \hat{b} \sqrt{\frac{B}{AB - C^2}} < u < \hat{u} + \zeta \hat{b} \sqrt{\frac{B}{AB - C^2}} . \tag{8.5.4}$$

Denote these confidence limits by b_{AL}, b_{AU}, u_{AL} and u_{AU} such that $b_{AL} < b < b_{AU}$ and $u_{AL} < u < u_{AU}$.

8.5.2 Intervals Based on Likelihood Score Corrected for Bias and Skewness (BI)

Consider the log likelihood (8.2.3). Define

$$I_{uu} = - E\left(\frac{\partial^2 l}{\partial u^2} \right); \; I_{ub} = - E\left(\frac{\partial^2 l}{\partial u \partial b} \right); \; I_{bb} = - E\left(\frac{\partial^2 l}{\partial b^2} \right)$$

and $I_{uu.b} = I_{uu} - I^2_{ub} / I_{bb}$. Suppose the scale parameter b is of interest and the location parameter u is treated as a nuisance parameter. Then following the procedure described in section 7.5.2, the adjusted score statistic is

$$T_b = a\, \partial l/\partial b + \partial l/\partial u, \tag{8.5.5}$$

where $a = (I_{bb.u})^{-1/2}$ and $c = -\, I_{ub}\, (I_{bb})^{-1}\, (I_{bb.u})^{-1/2}$ has asymptotically standard normal distribution. Thus, the approximate $100(1-\alpha)\%$ confidence interval for b can be obtained by solving

$T_b = \pm\, \zeta,$

where ζ is as defined in section 8.5.1. Now, to the order $O(n^{-1})$, the bias of the statistic T_b is

$$B(T_b) = -\frac{1}{2\, I_{uu}} \left\{ \begin{array}{l} a\left[E\left(\dfrac{\partial^3 l}{\partial b \partial u^2} \right) + 2\, \dfrac{\partial I_{ub}}{\partial u} \right] \\ + c\left[E\left(\dfrac{\partial^3 l}{\partial u^3} \right) + 2\, \dfrac{\partial I_{uu}}{\partial u} \right] \end{array} \right\}.$$

The third cumulant of T_b, is given by

$$K_3(b) = \left\{ \begin{array}{c} 2\, E\left(\dfrac{\partial^3 l}{\partial b^3} \right) + 3\, \dfrac{\partial I_{bb}}{\partial b} \\ -\, 3\, \dfrac{C}{A} \left[2\, E\left(\dfrac{\partial^3 l}{\partial u \partial b^2} \right) + 2\, \dfrac{\partial I_{ub}}{\partial b} + \dfrac{\partial I_{bb}}{\partial u} \right] \\ +\, 3\, \left(\dfrac{C}{A}\right)^2 \left[2\, E\left(\dfrac{\partial^3 l}{\partial b \partial u^2} \right) + \dfrac{\partial I_{uu}}{\partial b} + 2\, \dfrac{\partial I_{ub}}{\partial u} \right] \\ -\, \left(\dfrac{C}{A}\right)^3 \left[2\, E\left(\dfrac{\partial^3 l}{\partial u^3} \right) + 3\, \dfrac{\partial I_{uu}}{\partial u} \right] \end{array} \right\} (I_{bb.u})^{-3/2}.$$

In our context, we have $I_{bb.u} = (B - C^2/A)/b^2$,

$$\frac{\partial l}{\partial u} = \frac{1}{b}\left\{\sum_{i=1}^{n} \exp\left(\frac{y_i - u}{b}\right) - r\right\},$$

$$\frac{\partial l}{\partial b} = \frac{1}{b}\left\{\sum_{i=1}^{n} \left(\frac{Y_i - u}{b}\right)\exp\left(\frac{Y_i - u}{b}\right) - \sum_{i=1}^{r} \left(\frac{Y_i - u}{b}\right) - r\right\},$$

$$\frac{\partial I_{ub}}{\partial u} = -\frac{1}{b^3}\sum_{i=1}^{n} (1 + k_i)\, e^{k_i - e^{k_i}},$$

$$\frac{\partial I_{uu}}{\partial u} = -\frac{1}{b^3}\sum_{i=1}^{n} e^{k_i - e^{k_i}},$$

$$\frac{\partial I_{bb}}{\partial u} = -\frac{1}{b^3}\sum_{i=1}^{n} (k_i + 1)^2\, e^{k_i - e^{k_i}},$$

$$\frac{\partial I_{ub}}{\partial b} = -\frac{1}{b^3}\left\{2C + \sum_{i=1}^{n} k_i\,(k_i + 1)\, e^{k_i - e^{k_i}}\right\},$$

$$\frac{\partial I_{bb}}{\partial b} = -\frac{1}{b^3}\left\{2B + \sum_{i=1}^{n} k_i\,(k_i + 1)^2\, e^{k_i - e^{k_i}}\right\},$$

$$\frac{\partial I_{uu}}{\partial b} = -\frac{1}{b^3}\left\{ 2A + \sum_{i=1}^{n} k_i \; e^{k_i - e^{k_i}} \right\},$$

$$E\left(\frac{\partial^3 l}{\partial u^2 \partial b}\right) = \frac{1}{b^3}\,(2A + C),$$

$$E\left(\frac{\partial^3 l}{\partial u^3}\right) = \frac{A}{b^3},$$

$$E\left(\frac{\partial^3 l}{\partial u \partial b^2}\right) = \frac{1}{b^3}\left\{ 4C + \sum_{i=1}^{n} \left(\begin{array}{c} \int_0^{e^{k_i}} V(\log V)^2 e^{-V}\, dV \\ + k_i^2 \; e^{k_i - e^{k_i}} \end{array} \right) \right\},$$

and

$$E\left(\frac{\partial^3 l}{\partial b^3}\right) = \frac{1}{b^3}\sum_{i=1}^{n} \left\{ \begin{array}{c} \int_0^{e^{k_i}} V(\log V)^3 e^{-V}\, dV + k_i^3 \; e^{k_i - e^{k_i}} \\ + 6 \int_0^{e^{k_i}} V(\log V)^2 e^{-V}\, dV + k_i^2 \; e^{k_i - e^{k_i}} \\ + 4\left(1 - e^{-e^{k_i}}\right) \end{array} \right\}.$$

Denote the expression above for $b^3 E(\partial^3 l/\partial b^3)$ by G. Then after some steps of algebra, we obtain

$$B(T_b) = -\frac{1}{A\sqrt{B - C^2/A}} \left\{ A + \sum_{i=1}^{n} \left(\frac{C}{A} - 1 - k_i \right) e^{k_i - e^{k_i}} \right\}, \tag{8.5.6}$$

and

$$K_3(b) = (2F + 3H)/6\,(I_{bb.u})^{3/2}, \tag{8.5.7}$$

where $F = G - 3B - 3C(E + C)/A + 2C(C/A)^2$,

$$H = \sum_{i=1}^{n} \left\{ \left(\frac{C}{A}\right)^3 - \left(\frac{C}{A}\right)^2 (3k_i + 2) + (3k_i^2 + 4k_i +)\frac{C}{A} \right\} e^{k_i - e^{k_i}}$$

and

$$E = \sum_{i=1}^{n} \left\{ \int_0^{e^{k_i}} V(\log V)^2 e^{-V} dV + k_i^2 \; e^{k_i - e^{k_i}} \right\}.$$

Now, the score statistic T_b corrected for bias and skewness given by $T_b - B(T_b) - K_3(b)(\zeta^2 - 1)/6$ is better approximated by $N(0,1)$. The approximate $100(1-\alpha)\%$ confidence interval for b is then obtained by solving

$$T_b - B(T_b) - \frac{K_3(b)}{6}(\zeta^2 - 1) = \pm\zeta. \tag{8.5.8}$$

Equation (8.5.8) involves the nuisance parameter u which is replaced by its MLE

$$\hat{u} = \hat{u}(b) = b \log\left(\frac{1}{r} \sum_{i=1}^{n} \exp(Y_i/b) \right)$$

for a given value of b. Then the equation (8.5.8) can be solved for b iteratively. The solutions provide the approximate 100(1-α)% lower and upper confidence limits for b. Denote these limits by b_{BL} and b_{BU} such that $b_{BL} < b < b_{BU}$.

For the construction of confidence interval for u, we define the score statistic

$$T_u = a' \frac{\partial l}{\partial u} + c' \frac{\partial l}{\partial b}, \tag{8.5.9}$$

where $a' = (I_{uu.b})^{-1/2}$ and $c' = - I_{ub} (I_{bb})^{-1} (I_{uu.b})^{-1/2}$. Following the procedure discussed in section 7.5.2, we obtain the bias of T_u as

$$B(T_u) = -\frac{1}{2B}\left\{E + 4C - 2P + (2Q - G)\frac{C}{B}\right\}(I_{uu.b})^{-1/2}, \tag{8.5.10}$$

and the third cumulant of T_u as

$$K_3(u) = (2W + 3W') / 6 (I_{uu.b})^{-3/2}, \tag{8.5.11}$$

where

$$P = \sum_{i=1}^{n} k_i (k_i + 1) e^{k_i - e^{k_i}},$$

$$Q = \sum_{i=1}^{n} k_i (k_i + 1)^2 e^{k_i - e^{k_i}},$$

$$W = A - 3(A + C)\frac{C}{B} + 3(F + 3C)\left(\frac{C}{B}\right)^2 - G\left(\frac{C}{B}\right)^3$$

and

$$W' = \sum_{i=1}^{n} \left\{ \begin{array}{c} k_i\,(k_i + 1)^3\,(\frac{C}{B})^3 - (3\,k_i^2 + 4\,k_i + 1)\,(\frac{C}{B})^2 \\ + (3\,k_i + 2)\,\frac{C}{B} - 1 \end{array} \right\} e^{k_i - e^{k_i}} .$$

Following the discussion in section 8.5.1, the approximate 100(1-α)% confidence interval for u is obtained by solving

$$T_u - B(T_u) - \frac{K_3(u)}{6}(\zeta^2 - 1) = \pm\,\zeta\,. \tag{8.5.12}$$

We can see that the equation (8.5.12) depends on the nuisance parameter b which can be replaced by its MLE $\tilde{b} = \tilde{b}(u)$ for given u. This can be obtained by setting $\partial l/\partial b = 0$; that is

$$\sum_{i=1}^{n}\left(\frac{Y_i - u}{b}\right)\exp\left(\frac{Y_i - u}{b}\right) - \sum_{i=1}^{r}\left(\frac{Y_i - u}{b}\right) - r = 0. \tag{8.5.13}$$

However, no explicit solution for $\tilde{b} = \tilde{b}(u)$ is available from (8.5.13). Therefore, for the construction of approximate 100(1-α)% confidence interval for the parameter u, we need to solve equations (8.5.12) and (8.5.13) simultaneously. Denote the solutions by u_{BL} and u_{BU} such that $u_{BL} < u < u_{BU}$.

8.5.3 Intervals Based on Likelihood Ratio (LI)

Again we here follow the procedure discussed in section 7.5.3. Consider the log likelihood function $l(u,b)$ as given in (8.2.3). Suppose the scale parameter b is of interest and the location parameter u is treated as a nuisance parameter. Then, following section 2.16.3, the log likelihood ratio statistic for testing b is given by

$$LR_b = 2\left\{ r \log (b/\hat{b}) + \sum_{i=1}^{r} \left(\frac{Y_i - \hat{u}}{\hat{b}} - \frac{Y_i - \bar{u}}{b} \right) \right\}, \tag{8.5.14}$$

which is approximately distributed as chi- squared with one degree of freedom, where

$$\bar{u} = \bar{u}(b) = b \log \left\{ \frac{1}{r} \sum_{i=1}^{n} \exp(Y_i / b) \right\}. \tag{8.5.15}$$

Expression (8.5.14) cannot be manipulated analytically to obtain explicit confidence limits for b. The simplest way to determine such limits is obtained from $LR_b \le \chi^2_{(1-\alpha)}(1)$. Thus, the b values that satisfy the equation

$LR_b = \chi^2_{(1-\alpha)}(1)$ are the approximate $100(1-\alpha)\%$ confidence limits for b. Denote these limits by b_{LL} and b_{LU} such that $b_{LL} < b < b_{LU}$.

Similarly, for the construction of the confidence interval for u, we obtain the log likelihood ratio statistic LR_u as

$$LR_u = 2\left\{ r \log (\bar{b}/\hat{b}) + \sum_{i=1}^{r} \left(\frac{Y_i - \hat{u}}{\hat{b}} - \frac{y_i - u}{\bar{b}} \right) - r + \sum_{i=1}^{n} \exp\left(\frac{Y_i - u}{\bar{b}} \right) \right\} \tag{8.5.16}$$

and

$$\left\{ \sum_{i=1}^{n} \left(\frac{Y_i - u}{\bar{b}} \right) \exp\left(\frac{Y_i - u}{\bar{b}} \right) - \sum_{i=1}^{r} \left(\frac{Y_i - u}{\bar{b}} \right) - r \right\} = 0\ . \tag{8.5.17}$$

From (8.5.17), $\bar{b} = \bar{b}(u)$ cannot be given explicitly as a function of u. However, solving

$$LR_u = \chi^2_{(1-\alpha)}(1), \tag{8.5.18}$$

subject to (8.5.17) yields the approximate 100(1-α)% confidence limits for u. The solutions are denoted by u_{LL} and u_{LU} such that $u_{LL} < u < u_{LU}$.

8.5.4 Intervals Based on Adjusted Likelihood Ratio (DI)

Suppose that the log likelihood function in terms of u and b is given by (8.2.3). As stated in section 7.5.4, we define

$V_1 = (u - \hat{u})/\hat{b}$, $V_2 = \log(b/\hat{b})$ and $A_i = (y_i - \hat{u})/\hat{b}$, $i = 1,\ldots,n$. Then the log likelihood (8.2.3) reduces to

$$l(V_1,V_2) = - r\, V_2 + \sum_{i=1}^{r} P_i - \sum_{i=1}^{n} e^{P_i} , \tag{8.5.19}$$

where $P_i = (A_i - V_1)\exp(- V_2)$. Using the notations described in section 7.5.4, we define the likelihood ratio statistic (Λ_b) for testing the scale parameter b as $\Lambda_b = 2\ [\ l(0,0) - l(\tilde{V}_1,V_2)\ ]$, where $\tilde{V}_1 = \tilde{V}_1(V_2)$ is obtained from $\partial l/\partial V_1 = 0$. Now, the marginal tail probability of V_2 is, approximately, given by

$$P(\ V_2 \le v_2\ ) = \Phi(SR_b\ ) + \phi(SR_b\ )\ [S_b^*\ ] + O(n^{-3/2}\ ) , \tag{8.5.20}$$

where SR_b and S^*_b are as defined in section 7.5.4. In our context, we obtain

$$\Lambda_b = 2 \left[r\, V_2 + \sum_{i=1}^{r} (A_i - \tilde{P}_i\ ) \right],$$

$$\tilde{P}_i = (A_i - \tilde{V}_1\ ) \exp(-V_2\ ) ,$$

$$\tilde{V}_1 = e^{V_2} \log \left[\frac{1}{r} \sum_{i=1}^{n} \exp(A_i \, e^{-V_2}) \right],$$

$$l_b \; (\tilde{V}_1, V_2) = \sum_{i=1}^{n} \tilde{P}_i \, e^{\tilde{P}_i} - \sum_{i=1}^{r} \tilde{P}_i - r \, ,$$

$$- \, l_{u\,u} \, (\tilde{V}_1, V_2) = \left(\sum_{i=1}^{n} \exp(\tilde{P}_i \,) \right) e^{-2 V_2} \, ,$$

$$I^{0}_{v\,:\,u,u} = - \left. \frac{\partial^2 l}{\partial V_1 \, \partial V_1} \right|_{V=0} = r \, ,$$

$$I^{0}_{v\,:\,u,b} = - \left. \frac{\partial^2 l}{\partial V_1 \, \partial V_2} \right|_{V=0} = \sum_{i=1}^{n} A_i \; \exp(A_i \,)$$

and

$$I^{0}_{v\,:\,b,b} = - \left. \frac{\partial^2 l}{\partial V_2 \, \partial V_2} \right|_{V=0} = \sum_{i=1}^{n} A_i^2 \; e^{A_i} + r \, .$$

Now, the approximate 100(1-α)% lower and upper confidence limits V^b_L and V^b_U of V_2 can be obtained by setting the expression (8.5.20) to $\alpha/2$ and $(1-\alpha/2)$ respectively. Then, we can easily show that the approximate 100(1-α)% confidence interval for the parameter b is given by $\hat{b} \, V^b_L < b < \hat{b} \, V^b_U$. Denote these limits by b_{DL} and b_{DU} such that $b_{DL} < b < b_{DU}$. Following the same steps described above, for constructing confidence interval for u, the approximate marginal tail probability of the pivotal quantity V_1 is given as

$$P(V_1 \le v_1) = \Phi(SR_u) + \phi(SR_u) [S_u^*] + O(n^{-3/2}) , \tag{8.5.21}$$

where SR_u and S_u^* are as defined in section 7.5.4. The necessary terms for this expression are obtained as

$$SR_u = \begin{cases} -\sqrt{\Lambda_u} , & u < \hat{u} \\ \sqrt{\Lambda_u} , & u > \hat{u} \end{cases} ,$$

$$\Lambda_u = 2\left[r\,(\tilde{V}_2 - 1) + \sum_{i=1}^{r} (A_i - \tilde{P}_i) + \sum_{i=1}^{n} \exp(\tilde{P}_i) \right],$$

$$\tilde{P}_i = (A_i - V_1) \exp(- \tilde{V}_2) ,$$

$$l_u (V_1,\tilde{V}_2) = \left.\frac{\partial l}{\partial V_1}\right|_{V_2 = \tilde{V}_2} = \left(\sum_{i=1}^{n} e^{\tilde{P}_i} - r \right) \exp(- \tilde{V}_2) ,$$

$$- l_{bb} (V_1,\tilde{V}_2) = - \left.\frac{\partial^2 l}{\partial V_2\, \partial V_2}\right|_{V_2 = \tilde{V}_2} = \sum_{i=1}^{n} \tilde{P}_i\, e^{\tilde{P}_i} + r$$

and $\tilde{V} = \tilde{V}(V_1)$ is obtained from

$$\frac{\partial l}{\partial V_2} = \sum_{i=1}^{n} \tilde{P}_i e^{\tilde{P}_i} - \sum_{i=1}^{r} \tilde{P}_i - r = 0 . \tag{8.5.22}$$

Thus, the approximate 100(1-α)% lower and upper confidence limits V^u_L and V^u_U of V_1 can be obtained by setting the expression (8.5.21) to $\alpha/2$ and $(1-\alpha/2)$ respectively. Note that the expression (8.5.21) depends on the estimates $\tilde{V}_2$, which cannot be expressed in terms of V_1 explicitly from (8.5.22). Therefore, to obtain the limits stated above, we need to solve the appropriate equations with the equation (8.5.22) simultaneously. The approximate 100(1-α)% confidence interval for the parameter u is then expressed as $\hat{u} + \hat{b}\, V^u_L < u < \hat{u} + \hat{b}\, V^u_U$. Denote these limits by u_{DL} and u_{DU} such that $u_{DL} < u < u_{DU}$.

8.5.5 Simulation Study

We conduct a simulation study to determine and compare the coverage and tail probabilities of the confidence intervals for the parameters u and b based on the procedures AI, LI, DI and BI. The results are obtained based on 2000 samples generated through the IMSL subroutine RNWIB, and are displayed in Tables 8.1 and 8.2 at the nominal levels $\alpha = 0.10$ and 0.05. In order to restrict the number of parameters in the simulation study, we consider here only the case of common log censoring time $\eta_i = \eta$, $i = 1,\ldots,n$. The values of the parameters were chosen as $u = 0.15$ and $b = 0.9$. The censoring mechanism was defined by considering the reliability function of the extreme value distribution $R(y) = \exp(-\exp((y - u)/b))$, and the fixed censoring time η was taken by setting $R(\eta) = \pi$ for $\pi = 0.5, 0.25, 0.1$, where π represents the degree of censoring. Results based on uncensored situations, where $\pi = 0$, were also reported in Tables 8.1 and 8.2.

Results

Results for the parameter b reported in Table 8.1 indicate that the intervals BI, DI and LI provide desired coverage in almost all situations. The interval BI has tail probabilities closer to nominal even for small samples with heavy censoring. The procedure DI provides nearly symmetric tail probabilities under no censoring or light censoring. The performance of AI is inaccurate even for large sample sizes. The procedures LI and AI give asymmetric tail probabilities in all situations. The average interval lengths ordered from shortest to longest correspond to AI, LI, BI and DI.

From Table 8.2, which represents the results for the parameter u, we can see that the intervals BI, DI and LI have tail probabilities closer to nominal in most situations except for small sample sizes with heavy censoring. AI has asymmetric tail probabilities and smaller coverage than nominal for small to moderate sample sizes. The average lengths of all the four intervals become closer with increasing sample sizes.

8.6 INTERVAL ESTIMATION PROCEDURES FOR THE PARAMETERS OF EXTREME VALUE REGRESSION MODEL

8.6.1 Intervals Based on Asymptotic properties of the MLEs (AI)

As we mentioned in chapter 7 we do not consider this method in the regression situation because of its inadequate performance in case of two parameter distribution.

8.6.2 Intervals Based on Likelihood Score Corrected for Bias and Skewness (BI)

Consider the log likelihood function $l(\beta,b)$ given in (8.2.8). Suppose that the scale parameter b is of interest. Then following the steps discussed in section 7.6.2, using the same notation, the appropriate 100(1-α)% confidence interval for b can be obtained by solving

$$T_b - B(T_b) - \frac{K_3(b)\ (\zeta^2 - 1)}{6\ I_{bb.\beta}} = \pm\zeta\ \sqrt{I_{bb.\beta}}\ . \tag{8.6.1}$$

For s, t, q = 1,...,m, the required terms to compute the quantities T_b, $B(T_b)$, $K_3(b)$ and $I_{bb.\beta}$ are as follows:

$$\frac{\partial l}{\partial b} = \frac{1}{b}\left\{\sum_{i=1}^{n}\left(\frac{Y_i - X_i\ \beta}{b}\right)\exp\left(\frac{Y_i - X_i\ \beta}{b}\right) - \sum_{i=1}^{r}\left(\frac{Y_i - X_i\ \beta}{b}\right) - r\right\}, \tag{8.6.2}$$

$$\frac{\partial l}{\partial \beta_s} = \frac{1}{b}\left\{\sum_{i=1}^{n} X_i\ \exp\left(\frac{Y_i - X_i\ \beta}{b}\right) - \sum_{i=1}^{r} X_i\right\}, \tag{8.6.3}$$

$$V_{1i} = \int_0^{e^{k_i}} v\ \log(v)\ e^{-v}\ dv\ ,$$

$$V_{2i} = \int_0^{e^{k_i}} v\ (\log v)^2\ e^{-v}\ dv\ ,$$

$$V_{3i} = \int_0^{e^{k_i}} v\,(\log\, v)^3\, e^{-v}\, dv\ ,$$

$$b^3\, E\left(\frac{\partial^3 l}{\partial\beta_s \partial\beta_t \partial\beta_q} \right) = \sum_{i=1}^{n} X_{is}\, X_{it}\, X_{iq}\, (1 - e^{-e^{k_i}}) \ ,$$

$$b^3\, E\left(\frac{\partial^3 l}{\partial\beta_s \partial\beta_t \partial b} \right) = 2\, A_{st} + \sum_{i=1}^{n} X_{is}\, X_{it}\, (V_{1i} + k_i\, e^{k_i - e^{k_i}}) \ ,$$

$$b^3\, E\left(\frac{\partial^3 l}{\partial\beta_s \partial b^2} \right) = 4\, C_s + \sum_{i=1}^{n} X_{is}\, (V_{2i} + k_i^2\, e^{k_i - e^{k_i}}) \ ,$$

$$b^3\, \frac{\partial I_{\beta_s, \beta_t}}{\partial b} = - \sum_{i=1}^{n} X_{is}\, X_{it}\, (2 - 2\, e^{-e^{k_i}} + k_i\, e^{k_i - e^{k_i}}) \ ,$$

$$b^3\, \frac{\partial I_{\beta_s, b}}{\partial b} = - 2\, C_s - \sum_{i=1}^{n} X_{is}\, (k_i^2 + k_i)\, e^{k_i - e^{k_i}} \ ,$$

$$b^3 \frac{\partial I_{\beta_s, b}}{\partial \beta_t} = - \sum_{i=1}^{n} X_{is} X_{it} (k_i + 1) e^{k_i - e^{k_i}} ,$$

$$b^3 \frac{\partial I_{\beta_s, \beta_t}}{\partial \beta_q} = - \sum_{i=1}^{n} X_{is} X_{it} X_{iq} e^{k_i - e^{k_i}} ,$$

$$b^3 \frac{\partial I_{bb}}{\partial \beta_s} = - \sum_{i=1}^{n} X_{is} (k_i+1)^2 e^{k_i - e^{k_i}} ,$$

and

$$b^3 \frac{\partial I_{bb}}{\partial b} = - 2 B - \sum_{i=1}^{n} k_i (k_i+1)^2 e^{k_i - e^{k_i}} .$$

When m = 2, bias of T_b is obtained as

$$B(T_b) = \frac{A_{22} (M_1 - H_1) - 2 A_{12} (M_2 - H_2) + A_{11} (M_3 - H_3)}{2 [A_{11} A_{22} - (A_{12})^2] b} . \tag{8.6.4}$$

where

$$M_1 = f_1 \sum_{i=1}^{n} Q_{1i} + f_2 \sum_{i=1}^{n} X_i Q_{1i} ,$$

$$M_2 = f_1 \sum_{i=1}^{n} X_i Q_{1i} + f_2 \sum_{i=1}^{n} X_i^2 Q_{1i} ,$$

$$M_3 = f_1 \sum_{i=1}^{n} X_i^2 Q_{1i} + f_2 \sum_{i=1}^{n} X_i^3 Q_{1i} ,$$

$$H_1 = \sum_{i=1}^{n} Q_{2i} ,$$

$$H_2 = \sum_{i=1}^{n} X_i Q_{2i} ,$$

$$H_3 = \sum_{i=1}^{n} X_i^2 Q_{2i} ,$$

$f_1 = [A_{22} C_1 - A_{12} C_2]/[A_{11} A_{22} - (A_{12})^2]$,

$f_2 = [A_{11} C_2 - A_{12} C_1]/[A_{11} A_{22} - (A_{12})^2]$,

$$Q_{1i} = 1 - e^{-e^{k_i}} - 2 e^{k_i - e^{k_i}},$$

and

$$Q_{2i} = V_{1i} + 2\,(1 - e^{-e^{k_i}}) - (2 + k_i)\, e^{k_i - e^{k_i}} .$$

The third cumulant of T_b is

$$\begin{aligned} K_3(b) = {} & 2\,G - 3\,S - 3\,[\,f_1 \sum_{i=1}^{n} W_{1i} + f_2 \sum_{i=1}^{n} X_i\, W_{1i}\,] \\ & + 3\,[\,f_1^2 \sum_{i=1}^{n} W_{2i} + 2\,f_1\, f_2 \sum_{i=1}^{n} X_i\, W_{2i} + f_2^2 \sum_{i=1}^{n} X_i^2\, W_{2i}\,] \\ & - [\,f_1^3 \sum_{i=1}^{n} W_{3i} + f_2^3 \sum_{i=1}^{n} X_i^3\, W_{3i}\,] \\ & - 3\,f_1\, f_2\,[\,f_1 \sum_{i=1}^{n} X_i\, W_{3i} + f_2 \sum_{i=1}^{n} X_i^2\, W_{3i}\,]\,, \end{aligned} \tag{8.6.5}$$

where

$$S = 2\,B + \sum_{i=1}^{n} k_i\,(k_i+1)^2\, e^{k_i - e^{k_i}} ,$$

$$W_{1i} = 2\,V_{2i} + 4\,V_{1i} - (k_i^2+1)\, e^{k_i - e^{k_i}} ,$$

$$W_{2i} = Q_{2i} + V_{1i}$$

$$W_{3i} = 2\,(1 - e^{-e^{k_i}}) + 3\, e^{k_i - e^{k_i}}\,.$$

For the construction of confidence interval for a regression coefficient β_p, $p = 1,\ldots,m$, as we discussed in section 7.6.2, we need to solve the equation

$$T_\beta - B(T_\beta) - \frac{K_3(\beta)(\zeta^2 - 1)}{6\, I_{\beta\beta.b}} = \pm\, \zeta\, \sqrt{I_{\beta\beta.b}}\,. \tag{8.6.6}$$

When m = 2, we obtain

$$T_\beta = \frac{\partial l}{\partial \beta_2} - g_1 \frac{\partial l}{\partial \beta_1} - g_2 \frac{\partial l}{\partial b}\,,$$

$$B(T_\beta) = \frac{[\, B\,(M'_1 - H'_1) - 2\, C_1\,(M'_2 - H'_2) + A_{11}\,(M'_3 - H'_3)\,]}{2b\,(\,A_{11}\, B - C_1^2)}\,,$$

$$K_3(\beta) = \sum_{i=1}^{n} X_i^3\, W_{3i} - 3\, g_1 \sum_{i=1}^{n} X_i^2\, W_{3i} + 3 g_1^2 \sum_{i=1}^{n} X_i\, W_{3i} - g_1^3 \sum_{i=1}^{n} W_{3i}$$

$$- 3\, g_2 \sum_{i=1}^{n} X_i^2\, W_{2i} + 6\, g_1\, g_2 \sum_{i=1}^{n} X_i\, W_{2i} - 3\, g_1^2\, g_2 \sum_{i=1}^{n} W_{2i}$$

$$+ 3\, g_2^2 \sum_{i=1}^{n} X_i\, W_{1i} - 3\, g_1\, g_2^2 \sum_{i=1}^{n} W_{1i} - g_2^3\, (2\,G - 3\,S)\,,$$

where

$g_1 = (A_{12}\, B - C_1\, C_2)/(A_{11}\, B - C^2{}_1)$,

$g_2 = (A_{11}\, C_2 - A_{12}\, C_1)/(A_{11}\, B - C^2{}_1)$,

$$M'_1 = g_1\,(A_{11} - \sum_{i=1}^{n} e^{k_i - e^{k_i}}) + g_2\,[\,C_1 + 2\,(A_{11} - \sum_{i=1}^{n} (1+k_i)\,e^{k_i - e^{k_i}}\,)\,]\,,$$

$$M'_2 = g_1\,(C_1 - \sum_{i=1}^{n} (2k_i + 1)\,e^{k_i - e^{k_i}}\,)$$

$$+\,g_2\,[\,\sum_{i=1}^{n} (V_{2i} - (k_i^2 + 3\,k_i + 1)\,e^{k_i - e^{k_i}}\,)\,]\,,$$

$$M'_3 = g_1 \sum_{i=1}^{n} (V_{2i} - k_i\,(k_i + 2)\,e^{k_i - e^{k_i}}\,)$$

$$+\,g_2\,(\,G - 4\,B - 2 \sum_{i=1}^{n} k_i\,(k_i + 1)^2\,e^{k_i - e^{k_i}})\,,$$

$$H'_1 = A_{12} - 2 \sum_{i=1}^{n} X_i\,e^{k_i - e^{k_i}}\,,$$

$$H'_2 = C_2 - \sum_{i=1}^{n} X_i\,(2k_i + 1)\,e^{k_i - e^{k_i}}\,,$$

and

$$H'_3 = \sum_{i=1}^{n} X_i \left[V_{2i} - k_i (k_i+2)\, e^{k_i - e^{k_i}} \right].$$

8.6.3 Intervals Based on Likelihood Ratio (LI)

This procedure has been studied in various situations by many authors including Ostrouchov and Meeker (1988) and Doganaksoy and Scheme (1991). Consider the log likelihood function given in (8.2.8). Following the steps and notations described in section 7.6.3, the likelihood ratio statistic LR_b for testing the scale parameter b is obtained as

$$LR_b = 2 \left\{ r \log(b/\hat{b}) + \sum_{i=1}^{r} \left(\frac{Y_i - X_i \hat{\beta}}{\hat{b}} - \frac{Y_i - X_i \tilde{\beta}}{b} \right) \right\},$$

which is approximately distributed as chi-square with one degree of freedom. The expression for $\tilde{\beta}$ can not be obtained as a function of b explicitly from $\partial l / \partial \beta_s = 0$, s = 1,...,m, given in (8.6.3). Therefore, the LR based approximate 100(1-α)% confidence limits for the parameter b are obtained by solving the equation

$$LR_b = \chi^2_{(1-\alpha)}(1) \tag{8.6.7}$$

subject to the constraints $\partial l / \partial \beta_s = 0$, s = 1,... m. Denote the limits by b_{LL} and b_{LU} such that $b_{LL} < b < b_{LU}$.

For constructing the confidence interval for β_p, p = 1,...,m, the required likelihood ratio statistic LR_β is obtained as

$$LR_\beta = 2 \left\{ \begin{array}{c} r \log(\tilde{b}/\hat{b}) + \sum_{i=1}^{r} \left(\frac{Y_i - X_j \hat{\beta}}{\hat{b}} - \frac{Y_i - X_i \tilde{\beta}}{\tilde{b}} \right) \\ - r + \sum_{i=1}^{n} \exp\left(\frac{Y_i - X_i \tilde{\beta}}{\tilde{b}} \right) \end{array} \right\},$$

where $\tilde{\beta} = \tilde{\beta}(\beta_p)$ and $\tilde{b} = \tilde{b}(\beta_p)$ are the values of the nuisance parameters that maximize the log likelihood function for a given value of β_p. Because no closed form of these estimates $\tilde{\beta}$ and $\tilde{b}$ are available the approximate $100(1-\alpha)\%$ confidence limits for the regression parameter β_p , p = 1,...,m, need to be obtained by solving

$$LR_\beta = \chi^2_{(1-\alpha)}(1) \tag{8.6.8}$$

subject to $\partial l/\partial\beta_s = 0$, s = 1,...,m; s ≠ p and $\partial l/\partial b = 0$. Denote the limits by β_{LL} and β_{LU} such that $\beta_{LL} < \beta < \beta_{LU}$.

8.6.4 Intervals Based on Adjusted Likelihood Ratio (DI)

Following the procedure and notations described in section 7.6.4, we define $V_p = (\beta_p - \hat{\beta}_p)/\hat{b}$, p = 1,...,m, $V_{m+1} = \log(b/\hat{b})$ and $A_i = (Y_i - X_i\hat{\beta})/\hat{b}$, i = 1,...,n, where $\hat{\beta}$ and $\hat{b}$ are the MLEs of β and b respectively. Now, for given A_i, i = 1,...,n, the log likelihood function given in (8.2.8) can be written in terms of the pivotals $V_1,...,V_m,V_{m+1}$ as

$$l(V) = - r\, V_{m+1} + \sum_{i=1}^{r} P_i - \sum_{i=1}^{n} \exp(P_i) \,,$$

where, for i = 1,...,n,

$$P_i = (A_i - \sum_{s=1}^{m} X_{is} V_s) \exp(- V_{m+1})$$

and $V = (V_1,...,V_m,V_{m+1})'$. Suppose the parameter b is of interest. Then the related pivotal quantity is V_{m+1}. As discussed in section 7.6.4, the approximate marginal tail probability of the pivotal V_{m+1} is given by

$$P(V_{m+1} \le v_{m+1}) = \Phi(SR_{m+1}) + \phi(SR_{m+1})\,[S^*_{m+1}] + O(n^{-3/2}) , \qquad (8.6.9).$$

where Φ and ϕ are as defined in section 7.6.4. Equating the probability in (8.6.9) to $\alpha/2$ and $(1-\alpha/2)$ yield respectively the approximate $100(1-\alpha)\%$ lower and upper confidence limits for the pivotal V_{m+1}. Confidence limits for the parameter b are then obtained from the limits of V_{m+1} as discussed in section 8.5.4. In case of simple linear regression (m = 2), we obtain

$$LR_3 = 2\ [\ r\ V_i - \sum_{i=1}^{r} (\tilde{P}_i - A_i)\ ] ,$$

where $\tilde{P}_i = (A_i - \tilde{V}_1 - X_i\tilde{V}_2).\exp(-V_3)$. The elements of the observed information matrix I^0 and the matrix I^* are as follows:

$$I^0_{11} = - \left.\frac{\partial^2 l}{\partial V_1^2}\right|_{V=0} = r ,$$

$$I^0_{12} = - \left.\frac{\partial^2 l}{\partial V_1 \partial V_2}\right|_{V=0} = \sum_{i=1}^{r} X_i ,$$

$$I_{22}^{0} = - \left. \frac{\partial^2 l}{\partial V_2^2} \right|_{V=0} = \sum_{i=1}^{n} X_i^2 \exp(A_i) \ ,$$

$$I_{13}^{0} = - \left. \frac{\partial^2 l}{\partial V_1 \partial V_3} \right|_{V=0} = \sum_{i=1}^{r} A_i + r \ ,$$

$$I_{23}^{0} = - \left. \frac{\partial^2 l}{\partial V_2 \partial V_3} \right|_{V=0} = \sum_{i=1}^{n} X_i \, A_i \exp(A_i) \ ,$$

$$I_{33}^{0} = - \left. \frac{\partial^2 l}{\partial V_3^2} \right|_{V=0} = \sum_{i=1}^{n} A_i^2 \exp(A_i) + r \ ,$$

$$I_{11}^{*}(V_3) = - \left. \frac{\partial^2 l}{\partial V_1^2} \right|_{V=\tilde{V}(V_3)} = \left(\sum_{i=1}^{n} \exp(\tilde{P}_i) \right) \exp(-2V_3) \ ,$$

$$I_{12}^{*}(V_3) = - \left. \frac{\partial^2 l}{\partial V_1 \partial V_2} \right|_{V=\tilde{V}(V_3)} = \left(\sum_{i=1}^{n} X_i \exp(\tilde{P}_i) \right) \exp(-2V_3) \ ,$$

$$I_{22}^{*}(V_3) = - \left. \frac{\partial^2 l}{\partial V_2^2} \right|_{V=\tilde{V}(V_3)} = \left(\sum_{i=1}^{n} X_i^2 \exp(\tilde{P}_i) \right) \exp(-2V_3)$$

and

$$l_3(V(V_3)) = \left. \frac{\partial l}{\partial V_3} \right|_{V=\tilde{V}(V_3)} = \sum_{i=1}^{n} \tilde{P}_i \exp(\tilde{P}_i) - \sum_{i=1}^{r} \tilde{P}_i - r \ .$$

For constructing the confidence interval for a regression parameter β_p, p = 1,...,m, the appropriate marginal tail probability of the associated pivotal V_p is given by

$$P(V_p \le v_p) = \Phi(SR_p) + \phi(SR_p)\ [S_p^{*}\ ] + O(n^{-3/2}) \ , \tag{8.6.10}$$

where the terms SR_p, S^{*}_{p}, Φ and ϕ are as defined in section 7.6.4. Following the same steps described in section 7.6.4 we obtain the approximate 100(1-α)% confidence interval for the regression parameter from the marginal tail probability given in (8.6.10). Now, we consider the simple linear regression situation. Here, we provide the necessary terms to compute the required limits for the regression parameter β_2.

$$LR_2 = 2\ [\ r\ (\tilde{V}_3 - 1) + \sum_{i=1}^{r} (A_i - \tilde{P}_i) + \sum_{i=1}^{n} \exp(\tilde{P}_i)\ ] \ ,$$

$$\tilde{P}_i = (A_i - \tilde{V}_1 - V_2 X_i)\exp(-\tilde{V}_3) \ ,$$

$$l_2(V_2) = \frac{\partial l}{\partial V_2}\bigg|_{V=\tilde{V}(V_2)} = \left(\sum_{i=1}^{n} X_i \exp(\tilde{P}_i) - \sum_{i=1}^{r} X_i \right) \exp(-\tilde{V}_3) ,$$

$$I_{11}^{*}(V_2) = -\frac{\partial^2 l}{\partial V_1^2}\bigg|_{V=\tilde{V}(V_2)} = \left(\sum_{i=1}^{n} \exp(\tilde{P}_i) \right) \exp(-2\tilde{V}_3) ,$$

$$I_{12}^{*}(V_2) = -\frac{\partial^2 l}{\partial V_1 \partial V_3}\bigg|_{V=\tilde{V}(V_2)} = \left(\sum_{i=1}^{n} \tilde{P}_i + r \right) \exp(-\tilde{V}_3)$$

and

$$I_{22}^{*}(V_2) = -\frac{\partial^2 l}{\partial V_3^2}\bigg|_{V=\tilde{V}(V_2)} = \sum_{i=1}^{n} \tilde{P}_i^2 \exp(\tilde{P}_i) + r .$$

Table 8.1: Average lengths(AL), lower tail(L), upper tail(U) and coverage(C) probabilities(%) the confidence intervals AI, BI, LI and DI for b under time censoring. Based on 2000 samples.
u = 0.15; b = 0.9.

		$\alpha = 10.0$				$\alpha = 5.0$			
n,π		AL	L	U	C	AL	L	U	C
20,0.5	AI	0.877	0.3	12.9	86.8	1.045	0.0	9.8	90.2
	BI	1.053	4.9	5.6	89.5	1.297	2.7	2.9	94.4
	LI	0.942	3.7	6.3	90.0	1.148	1.5	2.8	95.7
	DI	1.245	7.4	2.4	90.2	1.568	4.4	1.3	94.3
20,0.25	AI	0.663	0.6	12.8	86.6	0.790	0.1	8.4	91.5
	BI	0.756	4.5	4.8	90.7	0.924	1.7	2.3	96.0
	LI	0.700	2.4	6.4	91.2	0.857	1.3	3.6	95.1
	DI	0.802	6.3	3.4	90.3	0.984	2.9	1.6	95.5
20,0.1	AI	0.567	0.8	12.1	87.1	0.675	0.1	8.6	91.2
	BI	0.637	4.1	4.4	91.5	0.779	2.0	2.8	95.2
	LI	0.592	2.7	7.1	90.1	0.720	1.3	3.3	95.4
	DI	0.656	5.1	3.6	91.3	0.799	2.5	1.7	95.8
20,0.0	AI	0.496	0.7	13.5	85.8	0.591	0.3	9.6	90.1
	BI	0.616	4.5	4.5	91.0	0.743	2.2	2.0	95.8
	LI	0.528	2.6	8.1	89.3	0.642	1.2	4.5	94.2
	DI	0.568	4.1	4.4	91.5	0.690	2.1	2.0	95.9
40,0.5	AI	0.617	1.0	10.3	88.7	0.735	0.1	7.7	92.2
	BI	0.674	4.6	5.7	89.7	0.814	2.1	2.5	95.4
	LI	0.642	2.9	7.1	90.0	0.783	1.4	3.5	95.1
	DI	0.716	6.4	3.7	89.9	0.872	3.8	1.2	95.0
40,0.25	AI	0.471	1.1	9.8	89.1	0.561	0.2	6.3	93.5
	BI	0.500	4.9	4.8	90.3	0.603	2.2	2.4	95.2
	LI	0.484	3.9	6.0	90.1	0.584	1.7	3.3	95.0
	DI	0.515	6.4	3.8	89.8	0.621	3.0	1.8	95.2
40,0.1	AI	0.406	1.6	9.6	88.8	0.483	0.2	5.8	94.0
	BI	0.428	4.6	4.4	91.0	0.516	2.0	2.4	95.6
	LI	0.415	3.6	5.8	90.6	0.500	1.8	3.1	95.1
	DI	0.435	5.4	4.0	90.6	0.524	2.4	2.2	95.4
40,0.0	AI	0.358	1.5	9.8	88.7	0.427	0.3	6.2	93.5
	BI	0.393	4.4	4.3	91.3	0.472	2.1	2.4	95.5
	LI	0.368	3.2	6.5	90.3	0.442	1.4	3.8	94.8
	DI	0.383	4.6	4.7	90.7	0.461	2.1	2.4	95.5

Table 8.2: Average lengths(AL), lower tail(L), upper tail(U) and coverage(C) probabilities(%) the confidence intervals AI, BI, LI and DI for u under time censoring. Based on 2000 samples.
u = 0.15; b = 0.9.

		$\alpha = 10.0$				$\alpha = 5.0$			
n,π		AL	L	U	C	AL	L	U	C
20,0.5	AI	1.109	0.0	10.8	89.2	1.321	0.0	7.8	92.2
	BI	1.455	4.6	4.2	91.2	1.856	2.8	2.3	94.9
	LI	1.253	3.2	6.7	90.1	1.569	1.7	3.0	95.2
	DI	1.586	6.5	3.5	90.0	2.014	3.2	1.9	94.9
20,0.25	AI	0.768	3.0	6.7	90.3	0.915	1.2	4.3	94.5
	BI	0.882	4.0	4.3	91.7	1.097	2.2	1.8	96.0
	LI	0.829	4.9	5.7	89.4	1.019	2.6	3.5	93.9
	DI	0.905	5.7	4.6	89.7	1.115	2.9	2.3	94.8
20,0.1	AI	0.697	5.1	6.4	88.5	0.830	3.2	4.0	92.8
	BI	0.760	4.7	4.5	90.8	0.939	1.7	1.8	96.5
	LI	0.734	5.0	5.9	89.1	0.894	2.8	3.5	93.7
	DI	0.777	5.1	4.8	90.1	0.948	2.8	2.5	94.7
20,0.0	AI	0.669	5.8	6.4	87.8	0.798	3.6	3.9	92.5
	BI	0.736	4.8	5.0	90.2	0.913	2.6	2.2	95.2
	LI	0.703	5.7	5.9	88.4	0.852	2.8	3.2	94.0
	DI	0.727	4.8	5.2	90.0	0.881	2.7	2.9	94.4
40,0.5	AI	0.761	0.6	7.8	91.6	0.907	0.1	4.5	95.4
	BI	0.847	5.1	4.0	90.9	1.035	2.8	2.0	95.2
	LI	0.814	4.0	4.5	91.5	0.989	2.0	2.3	95.7
	DI	0.881	6.4	3.6	90.0	1.076	3.6	1.8	94.6
40,0.25	AI	0.541	3.2	5.6	91.2	0.645	1.1	3.0	95.9
	BI	0.576	4.5	4.3	91.2	0.701	2.0	2.2	95.8
	LI	0.562	4.4	4.7	90.9	0.627	2.2	2.6	95.2
	DI	0.584	5.2	3.7	91.1	0.708	2.5	2.0	95.5
40,0.1	AI	0.498	5.0	5.1	89.9	0.593	2.4	2.8	94.8
	BI	0.518	4.4	4.6	91.0	0.628	2.1	2.5	95.4
	LI	0.512	4.7	5.0	90.3	0.623	2.2	2.6	95.2
	DI	0.525	4.9	4.2	90.9	0.632	2.3	2.4	95.3
40,0.0	AI	0.484	5.2	4.8	90.0	0.576	2.5	2.8	94.7
	BI	0.502	4.5	4.1	91.4	0.609	2.0	2.4	95.6
	LI	0.494	4.2	4.8	91.0	0.592	2.0	2.8	95.2
	DI	0.503	4.3	4.1	91.6	0.605	2.0	2.4	95.6

CHAPTER 9

CONCLUSIONS AND FURTHER RESEARCH

This chapter attempts to review the conclusions made in this thesis and to suggest some problems for future development.

In chapter 3, five statistics LR, M, MB, CL and EP, for testing the equality of L (≥ 2) gamma scale parameters in presence of an unknown common shape parameter are developed. Of these the $C(\alpha)$ statistic CL, whose asymptotic distribution is known to be chi-square with (L-1) degrees of freedom, holds nominal level well and it is easy to use. The distribution of the statistic EP is not known for $L > 2$. When $L = 2$, this test is equivalent to a F test which also shows anti-conservative behaviour. The statistic LR, M and MB are in general liberal. Based on empirically calculated critical values, in general, either the $C(\alpha)$ statistic CL or the statistic EP are most powerful. Thus, based on the information given above, the $C(\alpha)$ statistic is recommended for use except the situation where $n^1 \leq ... \leq n_L$ and $\lambda_1 < ... < \lambda_L$, in which the statistic CL has least power and the statistic EP is most powerful. For situations for which $n_1 \geq ... \geq n_L$ and $\lambda_1 < ... < \lambda_L$, the statistic CL is most powerful and EP has least power. More investigations into this problem such as the improvement of the $C(\alpha)$ statistic when $n_1 \leq ... \leq n_L$ and $\lambda_1 < ... < \lambda_L$ needs to be carried out. Moreover, in comparing the gamma means the validity of the assumption of common shape parameter should be checked. For this purpose, a $C(\alpha)$ test is recommended based on empirical level and power presented in chapter 3. However, in practice, the assumption of common shape parameter may not be true. In this situation for L = 2, the test statistic proposed by Shiue, Bain and Engelhardt (1988) is preferable. When L > 2, large sample procedures for this test need to be developed. Although the gamma model has been found to be useful in many areas of application including life testing problems, large sample statistical procedures for this distribution involving censored data remain unsolved.

In chapter 4, for the comparison of the mean life times of several two parameter exponential populations under failure censoring, a number of procedures have been developed. Of these a modified marginal likelihood ratio statistic (MB) is recommended. Procedures need to be developed for time censored data which is common in practice.

In chapter 5, for the comparison of several extreme value location parameters, first the validity of the assumption of common shape parameter should be checked. For this a modified likelihood ratio statistic is recommended. When the assumption of common scale parameter is valid, an adjusted $C(\alpha)$ statistic (ACLu) is recommended for the homogeneity test of extreme value location parameters. However, the assumption of common scale parameter may not be true in practice. It is therefore important to develop procedures for testing location parameters without the assumption of a common scale parameter. The procedures developed in this chapter are for failure censored data. Similar procedures need to be developed and studied for time censored data coming from extreme value distributions.

In chapters 7 and 8, for constructing confidence intervals for the parameters of extreme value distribution under both failure and time censoring, various procedures based on the likelihood are developed and studied through simulations. The procedures have been extended to construct confidence intervals for the parameters of the extreme value regression models. The performance of these procedures are currently under investigation.

REFERENCES

Bain, L. J. (1972). Inferences based on censored sampling from the Weibull or extreme value distribution. *Technometrics*, 14, 693 - 702.

Bain, L. J., and Engelhardt, M. (1981). Simple approximate distributional results for confidence and tolerance limits for the Weibull distribution based on maximum likelihood estimators. *Technometrics*, 23, 15 - 20.

Barndorff-Nielsen, O. E. (1986). Inference on full or partial parameters based on the standardized signed log likelihood ratio. *Biometrika*, 73, 307 - 322.

Barnwal, R. K., and Paul, S. R. (1988). Analysis of one-way layout of count data with negative binomial variation. *Biometrika*, 75, 215 - 222.

Bartholomew, D. J. (1963). The sampling distribution of an estimate arising in life testing. *Technometrics*, 5, 361 - 374.

Bartlett, M. S. (1937). Properties of sufficiency and statistical tests. *Proc. R. Soc. London A*, 160, 268 - 282.

Bartlett, M. S. (1953). Approximate confidence intervals I. *Biometrika*, 40, 12 - 19.

Bartlett, M. S. (1953). Approximate confidence intervals II, more than one unknown parameter. *Biometrika*, 40, 306 - 317.

Bartlett, M. S. (1955). Approximate confidence intervals III, a bias correction. *Biometrika*, 42, 201 - 204.

Bartoo, J. B., and Puri, P. S. (1967). On optimal asymptotic tests of composite hypothesis. *Ann. Math. Stat.*, 38, 1845 - 1852.

Bugaighis, M. M. (1988). Efficiencies of MLE and BLUE for parameters of an accelerated life test model. *IEEE Trans. Reliab.*, R37, 230 - 233.

Cohen, A. C. (1965). Maximum likelihood estimation in the Weibull distribution based on complete and censored samples. *Technometrics*, 7, 579 - 588.

Cox, D. R. (1962). Further results on tests of seperate families of hypotheses. *J. R. Stat. Soc. B*, 24, 406 - 423.

Cox, D. R., and Hinkley, D. R. (1974). *Theoretical Statistics*. London: Chapman and Hall.

Cox, D. R., and Reid, N. (1987). Parameter orthogonality and approximate conditional inference. *J. R. Stat. Soc. B*, 49, 1 - 39.

David, H. A. (1970). *Order Statistics*. New York: Wiley

Diciccio, T. J. (1984). On parameter transformations and interval estimation. *Biometrika*, 71, 477 - 485.

Diciccio, T. J. (1988). Likelihood inference for linear regression models. *Biometrika*, 75, 29 - 34.

Diciccio, T. J., Field, C. A., and Fraser, D. A. S. (1990). Approximations of marginal tail probabilities and inference for scalar parameters. *Biometrika*, 77, 77 - 95.

Doganaksoy, N., and Schmee, J. (1991). Comparisons of approximate confidence intervals for the smallest extreme value distribution simple linear regression model under time censoring. *Commun. Statist. - Simulation and Computation*, 20.

Engelhardt, M., and Bain, L. J. (1973). Some complete and censored sampling results for the Weibull or extreme value distribution. *Technometrics*, 15, 541 - 549.

Engelhardt, M., and Bain, L. J. (1974). Some results on point estimation for the two-parameter Weibull or extreme value distribution. *Technometrics*, 16, 49 - 56.

Engelhardt, M., and Bain, L. J. (1977). Simplified statistical procedures for the Weibull or extreme value distribution. *Technometrics*, 19, 323 - 331.

Epstein, B., and Tsao, C. K. (1953). Some tests based on ordered observations from two exponential populations. *Ann. Math. Statist.*, 24, 458 - 466.

Escobar, L. A., and Meeker, W. Q. (1986). Algorithm AS218: elements of the Fisher information matrix for the smallest extreme value distribution and censored data. *Appl. Stat.*, 35, 80 - 86.

Escobar, L. A., and Meeker, W. Q. (1991). Fisher information matrix for the extreme value, Normal, and logistic distributions and censored data.

Gastwirth, J. L., and Mahmoud, H. (1986). An efficiency robust non-parametric test for scale change for data from a gamma distribution. *Technometrics*, 28, 81 - 84.

Gupta, S. S., and Groll, P. S. (1961). Gamma distribution in acceptance sampling based on life tests. *J. Am. Stat. Assoc.*, 56, 942 - 970.

Harter, H. L., and Moore, H. (1965). Maximum likelihood estimation of the parameters of gamma and Weibull populations from complete and from censored samples. *Technometrics*, 7, 639 - 643.

Harter, H. L., and Moore, H. (1967). Asymptotic variances and covariances of maximum likelihood estimators, from censored samples, of the parameters of Weibull and gamma distributions. *Ann. Math. Stat.*, 38, 557 - 570.

Hogg, R. V., and Tanis, E. A. (1963). An iterative procedure for testing the equality of several exponential distributions. *J. Am. Stat. Assoc., 58, 435 - 443.*

Hsieh, h. K. (1981). On testing the equality of two exponential distributions. *Technometrics*, 23, 265 - 269.

Hsieh, H. K. (1986). An exact test for comparing location parameters of K exponential distributions with unequal scales based on Type II censored data. *Technometrics*, 28, 157 - 164.

IMSL (1987). *User's Manual Math/Library*. Version 1, IMSL Inc., Houston, Texas.

Johnson, N. L., and Kotz, S. (1970). *Continuous Univariate distributions, Vol. 1*, Boston, Massachusetts: Houghton Mifflin.

Kalbfleisch, J. D., and Sprott, D. A. (1970). Application of likelihood methods to models involving a large number of parameters (with discussion). *J. R. Stat. Soc. B*, 32, 175 - 208.

Kambo, N. S., and Awad, A. M. (1985). Testing equality of location parameters of K exponential distributions. *Commun. Statist. Theory and Method*, 14, 567 - 583.

Kendall, M. G., and Stuart, A. (1967). *The Advanced Theory of Statistics*, Vol. 2, 2nd ed. London: Griffin.

Khatri, C. G. (1981). Power of a test for location parameters of two exponential distributions. *Alig. J. Statist.*, 1, 8 - 12.

Kimball, F. (1946). Sufficient statistical estimation functions for the parameters of the distribution of maximum values. *Ann. Math. Statist.*, 17, 299 - 309.

Kumar, S., and Patel, H. I. (1971). A test for a comparison of two exponential distributions. *Technometrics*, 13, 183 - 189.

Lawless, J. F. (1976). Confidence interval estimation in the inverse power law model.

Appl. Stat., 25, 128 - 138.

Lawless, J. F. (1978). Confidence interval estimation for the Weibull and extreme value distributions. *Technometrics*, 20, 355 - 364.

Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*. New York: Wiley.

Lawless, J. F., and Mann, N. R. (1976). Tests for homogeneity for extreme value scale parameters. *Commun. Stat.*, A5, 389 - 405.

Levin, B., and Kong, F. (1990). Bartlett's bias correction to the profile score function is a saddlepoint correction. *Biometrika*, 77, 219 - 221.

Mann, N. R. (1967). Tables for obtaining the best linear invariant estimates of parameters of the Weibull distribution. *Technometrics*, 9, 629 - 645.

Mann, N. R., and Fertig, K. W. (1973). Tables for obtaining confidence bounds and tolerance bounds based on best linear invariant estimates of parameters of the extreme value distribution. *Technometrics*, 15, 87 - 101.

Mann, N. R., and Fertig, K. W. (1977). Efficient unbiased quantile estimators for moderate-size complete samples from extreme value and Weibull distributions: Confidence bounds and tolerance and prediction intervals. *Technometrics*, 19, 87 - 93.

Mann, N. R., Fertig, K. W., and Scheuer, E. M. (1971). Confidence and tolerance bounds and a new goodness of fit test for the two-parameter Weibull or extreme value distribution with tables for censored samples of size 3(1)25. Wright - Patterson Air force Base, Ohio, ARL Technical Report ARL 71 - 0077.

Mann, N. R., Schafer, R. E., and Singpurwalla, N. D. (1974). *Methods for Statistical Analysis of Reliability and Lifetime Data*. New York: Wiley.

McCool, J. I. (1970). Inferences on Weibull percentiles and shape parameter for maximum likelihood estimates. *IEEE Trans. Reliab.*, R19, 2 - 9.

McCool, J. I. (1974). Inferential techniques for Weibull populations. Wright - Patterson Air force Base, Ohio. ARL Technical Report ARL - 74 - 0180.

McCool, J. I. (1979). Analysis of single classification experiments based on censored samples from the two-parameter Weibull distribution. *J. Stat. Plan. Infer.*, 3, 39 - 68.

McCool, J. I. (1980). Confidence limits for Weibull regression with censored data. *IEEE Trans. Reliab.*, R29, 145 - 150.

McCool, J. I. (1982). Analysis of sets of two-parameter Weibull data arising in rolling contact endurance testing. Rolling contact fatique testing of bearing steels, ASTM, STP 771, J. J. C. Hoo, Ed., American Society for testing and materials, pp. 293 - 319.

McCullagh, P. (1984). Local Sufficiency. *Biometrika*, 71, 233 - 244.

McCullagh, P., and Nelder, J. A. (1989). *Generalized Linear Models*. London: Chapman and Hall.

Meeker, W. Q., and Nelson, W. (1974). Confidence limits for the Weibull distribution and singly censored data. General Electric Col. Corp. Research and Development TIS Report 74 CRD245.

Meeker, W. Q., and Nelson, W. (1976). Weibull percentiles estimates and confidence limits from singly censored data by maximum likelihood. *IEEE Trans. Reliab.*, R25, 20 - 24.

Meeker, W. Q., and Nelson, W. (1977). Weibull variances and confidence limits by maximum likelihood for singly censored data. *Technometrics*, 19, 473 - 476.

Mielke, P. W. J., and Johnson, E. S. (1974). Some generalized beta distributions of the second kind having desirable application features in hydrology and meteorology. *Water Resources Research*, 10, 223 - 226.

Moran, P. A. P. (1970). On asymptotically optimal test of composite hypothesis. *Biometrika*, 57, 47 - 55.

Moran, P. A. P. (1973). Asymptotic properties of homogeneity tests. *Biometrika*, 60, 79 - 85.

Nelson, W. (1970). Statistical methods for accelerated lifetest data - the inverse power law model. General Electric Co. Technical Report 71-C-011, Schenectady, New York.

Nelson, W. (1982). *Applied Life Data Analysis*. New York: Wiley.

Nelson, W., and Hahn, G. J. (1972). Linear estimation of a regression relationship from censored data. Part I - Simple methods and their applications. *Technometrics*, 14, 247 - 269.

Nelson, W., and Hahn, G. J. (1973). Linear estimation of a regression relationship from censored data. Part II - Best linear unbiased estimation and theory. *Technometrics*, 15, 133 - 150.

Nelson, W., and Meeker, W. Q. (1978). Theory of optimum accelerated censored life tests for Weibull and extreme value distributions. *Technometrics*, 20, 171 - 177.

Neyman, J. (1959). Optimal asymptotic tests for composite hypotheses. *Probability and Statistics*, Ed. U. Grenander, pp. 213 - 234, New York: Wiley.

Neyman, J., and Scott, E. (1965). Asymptotically optimal tests of composite hypothesis for randomized experiments with non controlled predictor variables. *J. Am. Stat. Assoc.*, 60, 699 - 721.

Ostrouchov, G., and Meeker, W. Q. (1988). Accuracy of approximate confidence bounds computed from interval censored Weibull and log normal data. *J. of Statist. Computation and Simulation*, 29, 43 - 76.

Paul, S. R. (1989). Testing for the equality of several correlation coefficients. *Canadian J. Statist.*, 17, 217 - 227.

Peers, H. W., and Iqbal, M. (1985). Asymptotic expansions for confidence limits in the presence of nuisance parameters, with applications. *J. R. Stat. Soc., B*, 47, 547 - 554.

Perng, S. K. (1978). A test for equality of two exponential distributions. *Statist. Neerlandica*, 32, 93 - 102.

Piegorsch, W. W. (1987). Performance of likelihood-based interval estimates for two-parameter exponential samples subject to Type I censoring. *Technometrics*, 29, 41 - 49.

Pike, M. C. (1966). A method of analysis of a certain class of experiments in carcinogenesis. *Biometrika*, 22, 142 - 161.

Rao, C. R. (1947). Large sample tests of statistical hypotheses concerning several parameters with application to problems of estimation. *Proc. Camb. Phil. Soc.*, 44, 50 - 57.

Sarhan, A. E, and Greenberg, B. G. (1962). *Contributions to Order Statistics*. New York: Wiley.

Shiue, W. K., and Bain, L. J. (1983). A Two sample test of equal gamma distribution scale parameters with unknown common shape parameter. *Technometrics*, 25, 377 - 381.

Shiue, W. K., Bain, L. J., and Engelhardt, M. (1988). Test of equal gamma distribution means with unknown and unequal shape parameters. *Technometrics*, 30, 169 - 174.

Simpson, J. (1972). Use of gamma distribution in single-cloud rainfall analysis. *Mon. Weather Rev.*, 100(4), 309 - 312.

Singh, N. (1983). The likelihood ratio test for the equality of location parameters of K ($\geq$ 2) exponential populations based on Type II censored samples. *Technometrics*, 25, 375 - 383.

Singh, N. (1985). A simple and asymptotically optimal test for the equality of K ($\geq$ 2) exponential distributions based on Type II censored samples. *Commun. Statist.-Theor. Meth.*, 14, 1615 - 1625.

Singh, N., and Narayan, P. (1983). The likelihood ratio test for the equality of K ($\geq$ 3) two parameter exponential distributions based on Type II censored samples. *J. Statist. Comput. Simul.*, 18, 287 - 297.

Singpurwalla, N. D. (1972). Extreme value from a lognormal law with applications to air pollution problems. *Technometrics*, 14, 703 - 711.

Sprott, D. A. (1973). Normal likelihoods and relation to a large sample theory of estimation. *Biometrika*, 60, 457 - 465.

Tarone, R. E. (1979). Testing the goodness of fit of the binomial distribution. *Biometrika*, 66, 585 - 590.

Tarone, R. E. (1985). On heterogeneity tests based on efficient scores. *Biometrika*, 72, 79 - 85.

Tiku, M. L. (1981). Testing equality of location parameters of two exponential distributions. *Alig. J. Statist.*, 1, 1 - 7.

Vander Weil, S. A., and Meeker, W. Q. (1990). Accuracy of approximate confidence bounds using censored Weibull regression data from accelerated life tests. *IEEE Trans. Reliab.*, R39, 346 - 351.

VITA AUCTORIS

The author was born in Vasavilan, Sri Lanka.

She worked as an Assistant Teacher at Christian College, Achuvely, Sri Lanka (1976 - 1980).

She received B. Sc. degree in Mathematics from the University of Jaffna, Sri Lanka in 1985.

She received Special degree in Statistics from the University of Jaffna, Sri Lanka in 1985.

She worked as an Assistant Lecturer at the Department of Mathematics and Statistics, University of Jaffna, Sri Lanka (1985 - 1986).

She received M. Sc. degree in Mathematics from the University of Windsor in 1988.